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by

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Summary. We consider a transient random walk (X_n) in random environment on a Galton–Watson tree. Under fairly general assumptions, we give a sharp and explicit criterion for the asymptotic speed to be positive. As a consequence, situations with zero speed are revealed to occur. In such cases, we prove that X_n is of order of magnitude n^{Λ}, with $\Lambda \in (0,1)$. We also show that the linearly edge reinforced random walk on a regular tree always has a positive asymptotic speed, which improves a recent result of Collevecchio [1].

Key words. Random walk in random environment, reinforced random walk, law of large numbers, Galton–Watson tree.

AMS subject classifications. 60K37, 60J80, 60F15.

1 Introduction

1.1 Random walk in random environment

Let ν be an $\mathbb{N}^*$-valued random variable (with $\mathbb{N}^* := \{1, 2, \cdots\}$) and $(A_i, i \geq 1)$ be a random variable taking values in $\mathbb{R}_+^{\mathbb{N}^*}$. Let $q_k := P(\nu = k)$, $k \in \mathbb{N}^*$. We assume $q_0 = 0$, $q_1 < 1$, and $m := \sum_{k\geq 0} kq_k < \infty$. Writing $V := (A_i, i \leq \nu)$, we construct a Galton–Watson tree as follows.

Let e be a point called the root. We pick a random variable $V(e) := (A(e_i), i \leq \nu(e))$ distributed as V, and draw $\nu(e)$ children to e. To each child e_i of e, we attach the random variable $A(e_i)$. Suppose that we are at the n-th generation. For each vertex x of the n-th generation, we pick independently a random vector $V(x) = (A(x_i), i \leq \nu(x))$ distributed as V, associate $\nu(x)$ children $(x_i, i \leq \nu(x))$ to x, and attach the random variable $A(x_i)$ to the child x_i. This leads to a Galton–Watson tree $\mathbb{T}$ of offspring distribution q, on which each vertex $x \neq e$ is marked with a random variable $A(x)$.

We denote by GW the distribution of $\mathbb{T}$. For any vertex $x \in \mathbb{T}$, let $\overleftarrow{x}$ be the parent of x and $|x|$ its generation ($|e| = 0$). In order to make the presentation easier, we artificially add a parent $\overleftarrow{e}$ to the root e. We define the environment ω by $\omega(\overleftarrow{e}, e) = 1$ and for any vertex $x \in \mathbb{T}\backslash\{\overleftarrow{e}\}$,

- $\omega(x, x_i) = \frac{A(x_i)}{1+\sum_{i=1}^{\nu(x)} A(x_i)}$, $\forall\, 1 \leq i \leq \nu(x)$,
- $\omega(x, \overleftarrow{x}) = \frac{1}{1+\sum_{i=1}^{b} A(x_i)}$.

For any vertex $y \in \mathbb{T}$, we define on $\mathbb{T}$ the Markov chain $(X_n,\, n \geq 0)$ starting from y by

$$\begin{aligned} P_\omega^y(X_0 = y) &= 1, \\ P_\omega^y(X_{n+1} = z \,|\, X_n = x) &= \omega(x, z)\,. \end{aligned}$$

Given $\mathbb{T}$, $(X_n,\, n \geq 0)$ is a $\mathbb{T}$-valued random walk in random environment (RWRE). We note from the construction that $\omega(x, .)$, $x \neq \overleftarrow{e}$ are independent.

Following [11], we also suppose that $A(x)$, $x \in \mathbb{T}$, $|x| \geq 1$, are identically distributed. Let A denote a random variable having the common distribution. We assume the existence of $\alpha > 0$ such that ess sup$(A) \leq \alpha$ and ess sup$(\frac{1}{A}) \leq \alpha$. The following criterion is known.

Theorem A (Lyons and Pemantle [11]) *The walk (X_n) is transient if $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$, and is recurrent otherwise.*

When $\mathbb{T}$ is a regular tree, Menshikov and Petritis [14] obtain the transience/recurrence criterion by means of a relationship between the RWRE and Mandelbrot's multiplicative cascades; Hu and Shi [8],[9] characterize different asymptotics of the walk in the recurrent case, revealing a wide range of regimes.

Throughout the paper, we assume that the walk is transient (i.e., $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$ according to Theorem A). Given the transience, natural questions arise concerning the rate of escape of the walk. The law of large numbers says that there exists a deterministic $v \geq 0$ (which can be zero) such that

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v, \qquad a.s.$$

This was proved by Gross [7] when $\mathbb{T}$ is a regular tree, and by Lyons et al. [13] when A is deterministic; their arguments can be easily extended in the general case (i.e., when $\mathbb{T}$ is a Galton–Watson tree and A is random).

We are interested in determining whether $v > 0$.

When A is deterministic, it is shown by Lyons et al. [13] that the transient random walk always has positive speed. Later, an interesting large deviation principle is obtained in Dembo et al. [3]. In the special case of non-biased random walk, Lyons et al. [12] succeed in computing the value of the speed.

We recall two results for RWRE on $\mathbb{Z}$ (which can be seen as a half line-tree). The first one gives a necessary and sufficient condition for RWRE to have positive asymptotic speed.

Theorem B (Solomon [16]) *If $\mathbb{T} = \mathbb{Z}$, then*

$$\mathbf{E}\left[\frac{1}{A}\right] < 1 \Longleftrightarrow \lim_{n\to\infty} \frac{X_n}{n} > 0 \quad a.s.$$

When the transient RWRE has zero speed, Kesten, Kozlov and Spitzer in [10] prove that the walk is of polynomial order. To this end, let $\kappa \in (0, 1]$ be such that $E\left[\frac{1}{A^\kappa}\right] = 1$. Under some mild conditions on A,

- if $\kappa < 1$, then $\frac{X_n}{n^\kappa}$ converges in distribution.
- If $\kappa = 1$, then $\frac{\ln(n)X_n}{n}$ converges in probability to a positive constant.

The aim of this paper is to study the behaviour of the transient random walk when $\mathbb{T}$ is a Galton–Watson tree. Let Leb represent the Lebesgue measure on $\mathbb{R}$ and let

$$\Lambda := Leb\left\{t \in \mathbb{R} : \mathbf{E}[A^t] \leq \frac{1}{q_1}\right\}. \tag{1.1}$$

If $q_1 = 0$, then we define $\Lambda := \infty$. Notice that this definition is similar to the definition of κ in the one-dimensional setting. Our first result, which is a (slightly weaker) analogue of Solomon's criterion for Galton–Watson tree $\mathbb{T}$, is stated as follows.

Theorem 1.1 *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$*, and let* Λ *be as in* (1.1).

(a) *If* $\Lambda < 1$*, the walk has zero speed.*

(b) *If* $\Lambda > 1$*, the walk has positive speed.*

Corollary 1.2 *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$*. If* $\mathbb{T}$ *is a regular tree, then the walk has positive speed.*

Theorem 1.1 extends Theorem B, except for the "critical case" $\Lambda = 1$.

Corollary 1.2 says there is no Kesten–Kozlov–Spitzer-type regime for RWRE when the tree is regular. Our next result exhibits such a regime for Galton–Watson trees $\mathbb{T}$.

Theorem 1.3 *Assume* $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{m}$*, and* $\Lambda \leq 1$*. Then*

$$\lim_{n\to\infty} \frac{\ln(|X_n|)}{\ln(n)} = \Lambda \quad a.s.$$

Since $\Lambda > 0$, the walk is proved to be of polynomial order. As expected, Λ plays the same role as κ.

1.2 Linearly edge reinforced random walk

The reinforced random walk is a model of random walk introduced by Coppersmith and Diaconis [2] where the particle tends to jump to familiar vertices. We consider the case where the graph is a b-ary tree $\mathbb{T}$, that is a tree where each vertex has b children ($b \geq 2$). At each edge (x, y), we initially assign the weight $\pi(x, y) = 1$. If we know the weights and the position of the walk at time n, we choose an edge emanating from X_n with probability proportional to its weight. The weight of the edge crossed by the walk then increases by a constant $\delta > 0$. This process is called the Linearly Edge Reinforced Random Walk (LERRW). Pemantle in [15] proves that there exists a real δ_0 such that the LERRW is transient if $\delta < \delta_0$ and recurrent if $\delta > \delta_0$ ($\delta_0 = 4, 29..$ for the binary tree). We focus, from now on, on the case $\delta = 1$, so that the LERRW almost surely is transient. Recently, Collevecchio in [1] shows that when $b \geq 70$ the LERRW has a positive speed v which verifies $0 < v \leq \frac{b}{b+2}$. We propose to extend the positivity of the speed to any $b \geq 2$.

Theorem 1.4 *The linearly edge reinforced random walk on a* b*-ary tree has positive speed.*

We rely on a correspondence between RWRE and LERRW, explained in [15]. By means of a Polya's urn model, Pemantle shows that the LERRW has the distribution of a certain RWRE, such that for any $y \neq \overleftarrow{e}$, the density of $\omega(y,z)$ on $(0,1)$ is given by

- $f_0(x) = \frac{b}{2}\,(1-x)^{\frac{b}{2}-1}$ if $z = \overleftarrow{y}$,
- $f_1(x) = \frac{\Gamma(\frac{b}{2}+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{b+1}{2})}\,x^{-\frac{1}{2}}(1-x)^{\frac{b-1}{2}}$ if z is a child of y.

Consequently, we only have to prove the positivity of the speed of this RWRE.

With the notation of Section 1.1, A is not bounded in this case, which means Theorem 1.1 does not apply. To overcome this difficulty, we prove the following result.

Theorem 1.5 *Let $\mathbb{T}$ be a b-ary tree and assume that $\inf_{[0,1]} \mathbf{E}[A^t] > \frac{1}{b}$ and*

$$E\left[\left(\sum_{i=1}^{b} A_i\right)^{-1}\right] < \infty\,.$$

Then the RWRE has positive speed.

Since the RWRE associated with the LERRW satisfies the assumptions of Theorem 1.5 as soon as $b \geq 3$, Theorem 1.4 follows immediately in the case $b \geq 3$. The case of the binary tree is dealt with separately.

The rest of the paper is organized as follows. We prove Theorem 1.5 in Section 2. In Section 3, we prove the upper bound in Theorem 1.3. Some technical results are presented in Section 4, and are useful in Section 5 in the proof of the lower bound in Theorem 1.3. In Section 6, we prove Theorem 1.1. The proof of Theorem 1.4 for the binary tree is the subject of Section 7. Finally, Section 8 is devoted to the computation of parameters used in the proof of Theorem 1.3.

2 The regular case, and the proof of Theorem 1.5

We begin the section by giving some notation. Let $\mathbf{P}$ denote the distribution of ω conditionally on $\mathbb{T}$, and $\mathbb{P}^x$ the law defined by $\mathbb{P}^x(\cdot) := \int P^x_\omega(\cdot)\mathbf{P}(\mathrm{d}\omega)$. We emphasize that P^x_ω, $\mathbf{P}$ and $\mathbb{P}^x$ depend on $\mathbb{T}$. We respectively associate the expectations E^x_ω, $\mathbf{E}$, $\mathbb{E}^x$. We denote also by

$\mathbf{Q}$ and $\mathbb{Q}^x$ the measures:

$$\begin{aligned}\mathbf{Q}(\cdot) &:= \int \mathbf{P}(\cdot) GW(\mathrm{d}\mathbb{T})\,,\\ \mathbb{Q}^x(\cdot) &:= \int \mathbb{P}^x(\cdot) GW(\mathrm{d}\mathbb{T})\,.\end{aligned}$$

For sake of brevity, we will write $\mathbb{P}$ and $\mathbb{Q}$ for $\mathbb{P}^e$ and $\mathbb{Q}^e$.

Define for $x, y \in \mathbb{T}$, and $n \geq 1$,

$$\begin{aligned}Z_n &:= \#\{x \in \mathbb{T} :\ |x| = n\}\,,\\ x \leq y &\Leftrightarrow \exists\, p \geq 0,\ \exists\, x = x_0, \ldots, x_p = y \in \mathbb{T} \text{ such that } \forall\, 0 \leq i < p\,,\ x_i = \overleftarrow{x}_{i+1}\,.\end{aligned}$$

If $x \leq y$, we denote by $[\![x, y]\!]$ the set $\{x_0, x_1, \ldots, x_p\}$, and say that $x < y$ if moreover $x \neq y$.

Define for $x \neq \overleftarrow{e}$, and $n \geq 1$,

$$\begin{aligned}T_x &:= \inf\{k \geq 0 :\ X_k = x\}\,,\\ T^*_x &:= \inf\{k \geq 1 :\ X_k = x\}\,,\\ \beta(x) &:= P^x_\omega(T_{\overleftarrow{x}} = \infty)\,.\end{aligned}$$

We observe that $\beta(x)$, $x \in \mathbb{T}\backslash\{\overleftarrow{e}\}$, are identically distributed under $\mathbf{Q}$. We denote by β a generic random variable distributed as $\beta(x)$. Since the walk is supposed transient, $\beta > 0$ $\mathbf{Q}$-almost surely, and in particular $E_{\mathbf{Q}}[\beta] > 0$.

We still consider a general Galton–Watson tree. We prove that the number of sites visited at a generation has a bounded expectation under $\mathbb{Q}$.

Lemma 2.1 *There exists a constant c_1 such that for any $n \geq 0$,*

$$E_{\mathbb{Q}}\left[\sum_{|x|=n} \mathbb{1}_{\{T_x < \infty\}}\right] \leq c_1\,.$$

Proof. By the Markov property, for any $n \geq 0$,

$$\sum_{|x|=n} P^e_\omega(T_x < \infty)\beta(x) = \sum_{|x|=n} P^e_\omega(T_x < \infty,\ X_k \neq \overleftarrow{x}\ \forall k > T_x) \leq 1\,.$$

The last inequality is due to the fact that there is at most one regeneration time at the n-th generation. Since $P^e_\omega(T_x < \infty)$ is independent of $\beta(x)$, we obtain:

$$1 \geq E_{\mathbf{Q}}\left[\sum_{|x|=n} P^e_\omega(T_x < \infty)\beta(x)\right] = \sum_{|x|=n} E_{\mathbf{Q}}\left[P^e_\omega(T_x < \infty)\right] E_{\mathbf{Q}}[\beta].$$

In view of the identity $E_{\mathbb{Q}}\left[\sum_{|x|=n} \mathbb{1}_{\{T_x<\infty\}}\right] = \sum_{|x|=n} E_{\mathbf{Q}}\left[P_\omega^e(T_x < \infty)\right]$, the lemma follows immediately. $\square$

Let us now deal with the case of the regular tree. We suppose in the rest of the section that there exists $b \geq 2$ such that $\nu(x) = b$ for any $x \in \mathbb{T} \setminus \{\overleftarrow{e}\}$.

Lemma 2.2 *If* $\mathbf{E}\left[\frac{1}{\sum_{i=1}^b A_i}\right] < \infty$*, then*

$$\mathbf{E}\left[\frac{1}{\beta}\right] < \infty\,.$$

Proof. Notice that $\mathbf{E}\left[\frac{1}{\max_{1\leq i\leq b} A_i}\right] < \infty$. For any $n \geq 0$, call v_n the vertex defined by iteration in the following way:

- $v_0 = e$
- $v_n \leq v_{n+1}$ and $A(v_{n+1}) = \max\{A(y),\, y \text{ is a child of } v_n\}$.

The Markov property tells that

$$\beta(x) = \sum_{i=1}^b \omega(x, x_i)\beta(x_i) + \sum_{i=1}^b \omega(x, x_i)(1 - \beta(x_i))\beta(x)\,,$$

from which it follows that for any vertex x,

$$\frac{1}{\beta(x)} = 1 + \frac{1}{\sum_{i=1}^b A(x_i)\beta(x_i)} \leq 1 + \min_{1\leq i\leq b} \frac{1}{A(x_i)\beta(x_i)}\,. \tag{2.1}$$

Let $\mathcal{C}(v_n) := \{y \text{ is a child of } v_n,\, y \neq v_{n+1}\}$ be the set of children of v_n different from v_{n+1}. Take $C > 0$ and define for any $n \geq 1$ the event

$$E_n := \{\forall k \in [0, n-1]\,, \forall y \in \mathcal{C}(v_k)\,, (A(y)\beta(y))^{-1} > C\}\,.$$

We extend the definition to $n = 0$ by $E_0^c := \emptyset$. Notice that the sequence of events is decreasing. Using equation (2.1) yields

$$\frac{\mathbb{1}_{E_n}}{\beta(v_n)} \leq (1 + C) + \frac{\mathbb{1}_{E_{n+1}}}{A(v_{n+1})\beta(v_{n+1})}\,. \tag{2.2}$$

On the other hand, by the i.i.d. property of the environment, we have

$$\mathbf{P}(E_n) = \mathbf{P}(E_1)^n\,.$$

By choosing C such that $\mathbf{P}(E_1) < 1$ and using the Borel–Cantelli lemma, we have $1\!\!1_{E_n} = 0$ from some $n_0 \geq 0$ almost surely. Iterate equation (2.2) to obtain

$$\frac{1}{\beta(e)} \leq (1+C)\left(1 + \sum_{n\geq 1} B(n)\right)$$

where $B(n) = 1\!\!1_{E_n} \prod_{k=1}^{n} \frac{1}{A(v_k)}$. Hence

$$\mathbf{E}\left[\frac{1}{\beta}\right] \leq (1+C)\left(1 + \sum_{n\geq 1} \mathbf{E}\left[B(n)\right]\right) .$$

We observe that $\mathbf{E}[B(n)] = \{\mathbf{E}\left[1\!\!1_{E_1} A(v_1)^{-1}\right]\}^n$. When C tends to infinity, $\mathbf{E}\left[1\!\!1_{E_1} A(v_1)^{-1}\right]$ tends to zero since $\mathbf{E}[A(v_1)^{-1}] < \infty$. Choose C such that $\mathbf{E}\left[1\!\!1_{E_1} A(v_1)^{-1}\right] < 1$ to complete the proof. □

For $x \in \mathbb{T}$ and $n \geq -1$, let

$$\begin{aligned} N(x) &:= \sum_{k\geq 0} 1\!\!1_{\{X_k = x\}} , \\ N_n &:= \sum_{|x|=n} N(x) , \\ \tau_n &:= \inf\{k \geq 0 \,:\, |X_k| = n\} . \end{aligned}$$

In words, $N(x)$ and N_n denote, respectively, the time spent by the walk at x and at the n-th generation, and τ_n stands for the first time the walk reaches the n-th generation. A consequence of the law of large numbers is that

$$\lim_{n\to\infty} \frac{\tau_n}{n} = \frac{1}{v} \quad \mathbb{Q}\text{-}a.s.$$

Our next result gives an upper bound for the expected value of N_n.

Proposition 2.3 *Suppose that $\mathbf{E}\left[\frac{1}{\sum_{i=1}^{b} A(x_i)}\right] < \infty$. There exists a constant c_2 such that for all $n \geq 0$, we have*

$$\mathbb{E}\left[\sum_{k=0}^{n} N_k\right] \leq c_2\, n .$$

Proof. By the strong Markov property, $P_\omega^x(N(x) = \ell) = \{P_\omega^x(T_x^* < \infty)\}^{\ell-1} P_\omega^x(T_x^* = \infty)$, for $\ell \geq 1$. Accordingly,

$$E_\omega^e\left[\sum_{k=0}^{n} N_k\right] = \sum_{0\leq |x|\leq n} P_\omega^e(T_x < \infty) E_\omega^x[N(x)] = \sum_{0\leq |x|\leq n} \frac{P_\omega^e(T_x < \infty)}{1 - P_\omega^x(T_x^* < \infty)} .$$

We observe that $1 - P_\omega^x(T_x^* < \infty) \ \geq \ \sum_{i=1}^b \omega(x, x_i)\, \beta(x_i)$. Since $P_\omega^e(T_x < \infty)$ is independent of $(\omega(x, x_i)\beta(x_i),\ 1 \leq i \leq b)$, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=0}^n N_k\right] &\leq \sum_{0 \leq |x| \leq n} \mathbf{E}\left[P_\omega^e(T_x < \infty)\right] \mathbf{E}\left[\left(\sum_{i=1}^b \omega(e, e_i)\beta(e_i)\right)^{-1}\right] \\
&= \mathbf{E}\left[\sum_{0 \leq |x| \leq n} P_\omega^e(T_x < \infty)\right] \mathbf{E}\left[\left(\sum_{i=1}^b \omega(e, e_i)\beta(e_i)\right)^{-1}\right]. \qquad (2.3)
\end{aligned}$$

Since $\sum_{i=1}^b \omega(e, e_i)\, \beta(e_i) \ \geq \ \{\min_{i=1\ldots b} \beta(e_i)\} \sum_{i=1}^b \omega(e, e_i)$, it follows that

$$\mathbb{E}\left[\sum_{k=0}^n N_k\right] \leq \mathbf{E}\left[\sum_{0 \leq |x| \leq n} P_\omega^e(T_x < \infty)\right] \mathbf{E}\left[\frac{1}{1 - \omega(e, \overleftarrow{e})}\right] \mathbf{E}\left[\left(\min_{i=1\ldots b} \beta(e_i)\right)^{-1}\right].$$

By definition, $\frac{1}{1-\omega(e,\overleftarrow{e})} = 1 + \frac{1}{\sum_{i=1}^b A(e_i)}$, which implies that $\mathbf{E}\left[\frac{1}{1-\omega(e,\overleftarrow{e})}\right] < \infty$. Notice also that $\mathbf{E}\left[(\min_{i=1\ldots b} \beta(e_i))^{-1}\right] \leq b\mathbf{E}[\frac{1}{\beta}] < \infty$ by Lemma 2.2. Finally, use Lemma 2.1 to complete the proof. $\square$

We are now able to prove the positivity of the speed.

Proof of Theorem 1.5. We note that $\tau_n \leq \sum_{k=-1}^n N_k$ and that $N_{-1} \leq N_0$. By Proposition 2.3, we have $\mathbb{E}[\tau_n] \leq 2c_2\, n$. Fatou's lemma yields that $\mathbb{E}[\liminf_{n\to\infty} \frac{\tau_n}{n}] \leq 2c_2$. Since $\lim_{n\to\infty} \frac{\tau_n}{n} = \frac{1}{v}$, then $v > 0$. $\square$

3 Proof of Theorem 1.3: upper bound

This section is devoted to the proof of the upper bound in Theorem 1.3, which is equivalent to the following:

Proposition 3.1 *We have*

$$\liminf_{n\to\infty} \frac{\ln(\tau_n)}{\ln(n)} \geq \frac{1}{\Lambda} \qquad \mathbb{Q} - a.s.$$

3.1 Basic facts about regenerative times

We recall some basic facts about regenerative times for the transient RWRE. These facts can be found in [7] in the case of regular trees, and in [13] in the case of biased random walks on Galton–Watson trees.

Let

$$D(x) := \inf\left\{k \geq 1 \,:\, X_{k-1} = x,\, X_k = \overleftarrow{x}\right\}, \quad (\inf \emptyset := \infty)\,.$$

We define the first regenerative time

$$\Gamma_1 := \inf\left\{k > 0 \,:\, \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\right\}$$

as the first time when the walk reaches a generation by a vertex having more than two children and never returns to its parent. We define by iteration

$$\Gamma_n := \inf\left\{k > \Gamma_{n-1} \,:\, \nu(X_k) \geq 2,\, D(X_k) = \infty,\, k = \tau_{|X_k|}\right\}$$

for any $n \geq 2$ and we denote by $\mathbb{S}(.)$ the conditional distribution $\mathbb{Q}(.\,|\,\nu(e) \geq 2,\, D(e) = \infty)$.

Fact *Assume that the walk is transient.*

(i) *For any* $n \geq 1,\ \Gamma_n < \infty \quad \mathbb{Q}$*-a.s.*

(ii) *Under* $\mathbb{Q}$, $(\Gamma_{n+1} - \Gamma_n, |X_{\Gamma_{n+1}}| - |X_{\Gamma_n}|)$, $n \geq 1$ *are independent and distributed as* $(\Gamma_1, |X_{\Gamma_1}|)$ *under the distribution* $\mathbb{S}$.

(iii) *We have* $E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,] < \infty$.

We feel free to omit the proofs of (i) and (ii), since they easily follow the lines in [7] and [13]. To prove (iii), we will show that $E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,] = 1/E_{\mathbf{Q}}[\beta]$. For any $n \geq 0$, we have, conditionally on $|X_{\Gamma_1}|$,

$$\mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| = n \,\middle|\, |X_{\Gamma_1}|\right) = \mathbb{1}_{\{|X_{\Gamma_1}| \leq n\}} \mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| - |X_{\Gamma_1}| = n - |X_{\Gamma_1}| \,\middle|\, |X_{\Gamma_1}|\right).$$

By the renewal theorem (see chapter XI of [6] for instance) and the fact that $\mathbb{1}_{\{|X_{\Gamma_1}| \leq n\}}$ tends to 1 $\mathbb{Q}$-almost surely, we obtain that

$$\lim_{n\to\infty} \mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| = n \,\middle|\, |X_{\Gamma_1}|\right) = 1/E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,]\,.$$

The dominated convergence yields then

$$\lim_{n\to\infty} \mathbb{Q}\left(\exists k \geq 2 \,:\, |X_{\Gamma_k}| = n\right) = 1/E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,]\,.$$

It remains to notice that on the other hand,

$$\mathbb{Q}\left(\exists k \in \mathbb{N} \,:\, |X_{\Gamma_k}| = n\right) = \mathbb{Q}\left(D(X_{\tau_n}) = \infty\right) = E_{\mathbf{Q}}[\beta]\,. \square$$

If we denote for any $n \geq 0$ by $u(n)$ the unique integer such that $\Gamma_{u(n)} \leq \tau_n < \Gamma_{u(n)+1}$, then Fact yields that $\lim_{n\to\infty} \frac{n}{u(n)} = E_{\mathbb{S}}[\,|X_{\Gamma_1}|\,]$. In turn, we deduce that

$$\liminf_{n\to\infty} \frac{\ln(\tau_n)}{\ln(n)} \geq \liminf_{n\to\infty} \frac{\ln(\Gamma_n)}{\ln(n)} \qquad \mathbb{Q}\text{-}a.s. \tag{3.1}$$

Let for $\lambda \in [0,1]$ and $n \geq 0$,

$$S(n,\lambda) := \sum_{k=1}^{n} (\Gamma_k - \Gamma_{k-1})^{\lambda} ,$$

by taking $\Gamma_0 := 0$. Then $(\Gamma_n)^{\lambda} \leq S(n,\lambda)$ since $\lambda \leq 1$, which gives, by the law of large numbers,

$$\limsup_{n\to\infty} \frac{(\Gamma_n)^{\lambda}}{n} \leq \lim_{n\to\infty} \frac{S(n,\lambda)}{n} = E_{\mathbb{S}}[\Gamma_1^{\lambda}] \qquad \mathbb{Q}\text{-}a.s. \tag{3.2}$$

3.2 Proof of Proposition 3.1

We construct a RWRE on the half-line as follows; suppose that $\mathbb{T} = \{-1, 0, 1, \ldots\}$. This would correspond to the case where $q_1 = 1$, $e = 0$, $\overleftarrow{e} = -1$. Marking each integer $i \geq 0$ with i.i.d. random variables $A(i)$, we thus define a one-dimensional RWRE as we defined it in the case of a Galton–Watson tree. We call $(R_n)_{n\geq 0}$ this RWRE. We still use the notation P_ω^i and $\mathbb{P}^i$ to name the quenched and the annealed distribution of (R_n) with $R_0 = i$. For $i \geq -1$ and $a \in \mathbb{R}_+$, define $T_i := \inf\{n \geq 0 \,:\, R_n = i\}$ and

$$p\,(i,a) := \mathbb{P}^0(T_{-1} \wedge T_i > a)\,, \tag{3.3}$$

where $b \wedge c := \min\{b,\, c\}$. We give two preliminary results.

Lemma 3.2 *Let Λ be as in (1.1). Then*

$$\liminf_{a\to\infty} \left\{ \sup_{i\geq 0} \frac{\ln\left(q_1^i p\,(i,a)\right)}{\ln(a)} \right\} \geq -\Lambda\,.$$

Proof. See Section 8. □

We return to our general RWRE $(X_n)_{n\geq 0}$ on a general Galton–Watson tree $\mathbb{T}$.

Lemma 3.3 *We have*

$$\liminf_{a\to\infty} \frac{\ln(\,\mathbb{S}\,(\Gamma_1 > a)\,)}{\ln(a)} \geq -\Lambda\,.$$

Proof. For any $x \in \mathbb{T}$, let $h(x)$ be the unique vertex such that

$$x \le h(x), \qquad \nu(h(x)) \ge 2\,, \qquad \forall\, y \in \mathbb{T},\, x \le y < h(x) \Rightarrow \nu(y) = 1\,.$$

In words, $h(x)$ is the oldest descendent of x such that $\nu(h(x)) \ge 2$ (and can be x itself if $\nu(x) \ge 2$). We observe that $\Gamma_1 \ge T_e^* \wedge T_{h(X_1)}$. Moreover, $\{\nu(e) \ge 2,\, D(e) = \infty\} \supset E_1 \cup E_2$ where

$$\begin{aligned} E_1 &:= \{\nu(e) \ge 2\} \cap \left\{X_1 \ne \overleftarrow{e},\, T_e^* < T_{h(X_1)},\, X_{T_e^*+1} \notin \{\overleftarrow{e},\, X_1\}\right\} \cap \{X_n \ne e,\, \forall\, n \ge T_e^* + 1\}\ , \\ E_2 &:= \{\nu(e) \ge 2\} \cap \left\{X_1 \ne \overleftarrow{e},\, T_{h(X_1)} < T_e^*\right\} \cap \left\{X_n \ne \overleftarrow{h(X_1)},\, \forall\, n \ge T_{h(X_1)} + 1\right\}\,. \end{aligned}$$

It follows that

$$\mathbb{S}(\Gamma_1 > a) \ge \frac{1}{\mathbb{Q}(\nu(e) \ge 2,\, D(e) = \infty)} (\mathbb{Q}(T_e^* > a,\, E_1) + \mathbb{Q}(T_{h(X_1)} > a,\, E_2))\,. \tag{3.4}$$

We claim that

$$\mathbb{Q}(T_e^* > a,\, E_1) = c_3\, \mathbb{Q}(T_{\overleftarrow{e}} < T_{h(e)},\, 1 + T_{\overleftarrow{e}} > a)\,. \tag{3.5}$$

Indeed, write

$$P_\omega^e(T_e^* > a,\, E_1) = \sum_{e_i \ne e_j} P_\omega^e\left(T_e^* < T_{h(e_i)},\, X_1 = e_i,\, X_{T_e^*+1} = e_j,\, D(e_j) = \infty,\, T_e^* > a\right).$$

By gradually applying the strong Markov property at times $T_e^* + 1$, T_e^* and at time 1, this yields

$$P_\omega^e(T_e^* > a,\, E_1) = \sum_{e_i \ne e_j} \omega(e, e_i) P_\omega^{e_i}\left(T_e < T_{h(e_i)},\, 1 + T_e > a\right) \omega(e, e_j)\beta(e_j).$$

Since $\omega(e, e_i)\omega(e_j)$, $\beta(e_j)$ and $P_\omega^{e_i}\left(T_e < T_{h(e_i)},\, 1 + T_e > a\right)$ are independent under $\mathbf{P}$, this leads to

$$\mathbb{P}(T_e^* > a,\, E_1) = \sum_{e_i \ne e_j} \mathbf{E}\left[\omega(e, e_i)\omega(e, e_j)\right] \mathbb{P}^{e_i}\left(T_e < T_{h(e_i)},\, 1 + T_e > a\right) \mathbf{E}\left[\beta(e_j)\right].$$

By the Galton–Watson property,

$$\mathbb{Q}(T_e^* > a,\, E_1) = E_{\mathbf{Q}}\left[\mathbb{I}_{\{\nu(e) \ge 2\}} \sum_{e_i \ne e_j} \omega(e, e_i)\omega(e, e_j)\right] \mathbb{Q}^e\left(T_{\overleftarrow{e}} < T_{h(e)},\, 1 + T_{\overleftarrow{e}} > a\right) E_{\mathbf{Q}}\left[\beta\right]\ ,$$

which gives (3.5). Similarly,

$$\mathbb{Q}(T_{h(X_1)} > a,\, E_2) = c_4 \mathbb{Q}\left(T_{\overleftarrow{e}} > T_{h(e)},\, 1 + T_{h(e)} > a\right) . \tag{3.6}$$

Finally, by (3.4), (3.5) and (3.6) we get

$$\mathbb{S}(\Gamma_1 > a) \geq c_5\, \mathbb{Q}\left(1 + T_{\overleftarrow{e}} \wedge T_{h(e)} > a\right) .$$

Conditionally on $|\,h(e)|$, the walk $|\,X_n|$, $0 \leq n \leq T_{\overleftarrow{e}} \wedge T_{h(e)}$ has the distribution of the walk R_n, $0 \leq n \leq T_{-1} \wedge T_{|h(e)|}$, as defined at the beginning of this section. For any $n \geq 0$, since $GW(|h(e)| = n) = q_1^n(1-q_1)$, it follows that $\mathbb{Q}\left(1 + T_{\overleftarrow{e}} \wedge T_{h(e)} > a\right) \geq q_1^n(1-q_1)p\,(n,a)$. Finally,

$$\liminf_{a\to\infty} \frac{\ln\left(\mathbb{S}\left(\Gamma_1 > a\right)\right)}{\ln(a)} \geq \liminf_{a\to\infty} \left\{ \sup_{n\geq 0} \frac{\ln\left(q_1^n p\,(n,a)\right)}{\ln(a)} \right\} .$$

Applying Lemma 3.2 completes the proof. □

We now have all of the ingredients needed for the proof of Proposition 3.1.

Proof of Proposition 3.1. If $\Lambda \geq 1$, Proposition 3.1 trivially holds since $\tau_n \geq n$. We suppose that $\Lambda < 1$, and let $\Lambda < \lambda < 1$. Let $M_n := \max\{\Gamma_k - \Gamma_{k-1},\ k = 2, \ldots n\}$. We have $\mathbb{Q}\left(M_n \leq n^{\frac{1}{\lambda}}\right) = \mathbb{Q}\left(\Gamma_2 - \Gamma_1 \leq n^{\frac{1}{\lambda}}\right)^n$. By Lemma 3.3, $\mathbb{Q}\left(\Gamma_2 - \Gamma_1 \leq n^{\frac{1}{\lambda}}\right) \leq 1 - n^{-1+\varepsilon}$ for some $\varepsilon > 0$ and large n. Consequently, $\sum_{n\geq 1} \mathbb{Q}\left(M_n \leq n^{\frac{1}{\lambda}}\right) < \infty$, and the Borel-Cantelli lemma tells that $\mathbb{Q}$-almost surely and for sufficiently large n, $M_n \geq n^{\frac{1}{\lambda}}$, which in turn implies that $\liminf_{n\to\infty} \frac{\Gamma_n - \Gamma_1}{n^{\frac{1}{\lambda}}} \geq 1$. We proved then that $\liminf_{n\to\infty} \frac{\ln(\Gamma_n)}{\ln(n)} \geq \frac{1}{\Lambda}$. Therefore, by equation (3.1),

$$\liminf_{n\to\infty} \frac{\ln(\tau_n)}{\ln(n)} \geq \frac{1}{\Lambda} \qquad \mathbb{Q}\text{-}a.s. \ \square$$

4 Technical results

We give, in this section, some tools needed in our proof of the lower bound in Theorem 1.3. Z_n stands as before for the size of the n-th generation of $\mathbb{T}$.

Lemma 4.1 *For every $b, n \geq 1$, we have*

$$E_{GW}[Z_n \mathbb{1}_{\{Z_n \leq b\}}] \leq b n^b q_1^{n-b} .$$

Proof. If $Z_n \le b$, then there are at most b vertices before the n-th generation having more than one child. Therefore,

$$GW(Z_n \le b) \le C_n^b q_1^{n-b} \le n^b q_1^{n-b}$$

and we conclude since $E_{GW}[Z_n \mathbb{1}_{\{Z_n \le b\}}] \le b\, GW(Z_n \le b)$. $\square$

Lemma 4.2 *Let β_i, $i \ge 1$ be independent random variables distributed as β. There exists $b_0 \ge 1$ such that*

$$E_{\mathbf{Q}}\left[\left(\frac{1}{\sum_{i=1}^{b_0} \beta_i}\right)^2\right] < \infty .$$

Proof. Let $\mathbb{T}^{(i)}$, $i \ge 1$ be independent Galton–Watson trees of distribution GW. We equip independently each $\mathbb{T}^{(i)}$ with an environment of distribution $\mathbf{P}$ so that we can look at the random variable $\beta(e^{(i)})$ where $e^{(i)}$ is the root of $\mathbb{T}^{(i)}$. Then $\beta(e^{(i)})$, $i \ge 1$ are independent random variables distributed as β.

Let $c_6 > 0$ be such that $\eta := \mathbf{Q}(\frac{1}{\beta} > c_6) < 1$. Recall that $\frac{1}{\alpha} \le A(x) \le \alpha$, $\forall x \in \mathbb{T}$, $\mathbb{Q}$-almost surely. Let $R^{(i)} := \inf\{n \ge 0 : \exists y \in \mathbb{T}^{(i)},\, |y| = n,\, \frac{1}{\beta(y)} \le c_6\}$ be the first generation in $\mathbb{T}^{(i)}$ where a vertex verifies $\frac{1}{\beta(y)} \le c_6$, and let $y^{(i)}$ be such a vertex y. Recall from equation (2.1) that

$$\frac{1}{\beta(x)} \le 1 + \frac{1}{A(x_j)\beta(x_j)}$$

for any child x_j of a vertex x. By iterating the inequality on the path $[\![e^{(i)}, y^{(i)}]\!]$, we obtain

$$\frac{1}{\beta(e^{(i)})} \le 1 + \sum_{z \in]\!]e, y^{(i)}[\![} H(z) + \frac{H(y^{(i)})}{\beta(y^{(i)})}$$

where $H(z) = \prod_{v \in]\!]e^{(i)}, z]\!]} \frac{1}{A(v)} \le \alpha^{|z|}$ for every $z \in \mathbb{T}$ by the bound assumption on A. Since $\frac{1}{\beta(y^{(i)})} \le c_6$, this implies

$$\frac{1}{\beta(e^{(i)})} \le c_7\, \alpha^{R^{(i)}} ,$$

for some constant c_7. There exist constants c_8 and c_9 such that for any $b \ge 1$,

$$\left(\frac{1}{\sum_{i=1}^{b} \beta(e^{(i)})}\right)^2 \le c_8\, c_9^{\min_{1 \le i \le b} R^{(i)}} . \tag{4.1}$$

We observe that

$$E_{\mathbf{Q}}\left[c_9^{\min_{1\le i\le b} R^{(i)}}\right] = \sum_{n=0}^{\infty} c_9^n\, \mathbf{Q}(\min_{1\le i\le b} R^{(i)} = n)$$

$$(4.2) \qquad \le \sum_{n=0}^{\infty} c_9^n\, \mathbf{Q}(R^{(1)} \ge n)^b\,.$$

We have, for any $n \ge 1$, $\mathbf{Q}(R^{(1)} \ge n) \le \mathbf{Q}\left(\forall |x| = n-1, \frac{1}{\beta(x)} > c_6\right)$. Recall that $\eta := \mathbf{Q}(\frac{1}{\beta} > c_6) < 1$. By independence,

$$\mathbf{Q}\left(\forall |x| = n-1, \frac{1}{\beta(x)} > c_6\right) = E_{GW}[\eta^{Z_{n-1}}]\,.$$

Let $q_1 < a < 1$. There exists a constant c_{10} such that $E_{GW}\left[\eta^{Z_\ell}\right] \le c_{10}\, a^{\ell+1}$ for any $\ell \ge 0$. Choose b_0 such that $c_9 a^{b_0} < 1$. Then by (4.2), $E_{\mathbb{Q}}\left[c_9^{\min_{1\le i\le b_0} R^{(i)}}\right] < \infty$, which completes the proof in view of (4.1). $\square$

Define for any $u, v \in \mathbb{T}$ such that $u \le v$ and for any $n \ge 1$:

$$(4.3) \qquad p_1(u,v) = P_\omega^u\left(T_{\overleftarrow{u}} = \infty\,,\, T_u^* = \infty\,,\, T_v = \infty\right)\,,$$

$$(4.4) \qquad \nu(u,n) = \#\{x \in \mathbb{T} \,:\, u \le x, |\, x - u| = n\}\,.$$

Lemma 4.3 *For all $n \ge 2$ and $k \in \{1,2\}$, we have*

$$(4.5) \qquad E_{\mathbf{Q}}\left[\sum_{|u|=n} \frac{1\!\!1_{\{Z_n > b_0\}}}{[p_1(e,u)]^k}\right] < \infty\,.$$

Proof. Let $n \ge 2$ and $k \in \{1,2\}$ be fixed integers and $\widetilde{n} := \inf\{\ell \ge 1 \;:\; Z_\ell > b_0\}$. Notice that $\{Z_n > b_0\} = \{\widetilde{n} \le n\}$. For any $u \in \mathbb{T}$ such that $|u| \ge \widetilde{n}$, let $\widetilde{u} \in \mathbb{T}$ be the unique vertex such that $|\widetilde{u}| = \widetilde{n}$ and $\widetilde{u} \le u$ that is the ancestor of u at generation $\widetilde{n}$. We have by the Markov property,

$$(4.6) \qquad p_1(e,u) \ge \sum_{|y|=\widetilde{n}-1} P_\omega^e(T_y < T_{e*}) P_\omega^y(T_{\overleftarrow{y}} = \infty\,,\, T_{\widetilde{u}} = \infty).$$

For any $|y| \le \widetilde{n}$ and y_i child of y, we observe that

$$\omega(y, y_i) = \frac{A(y_i)}{1 + \sum_{j=1}^{\nu(y)} A(y_j)} \ge \frac{1}{c_{11}\nu(y)}\,,$$

which is greater than $1/c_{11}b_0 := c_{12}$, by the boundedness assumption on A and the definition of $\widetilde{n}$. It yields that for any $|y| = \widetilde{n} - 1$,

$$P_\omega^e(T_y < T_e^*) \geq P_\omega^e(X_{\widetilde{n}-1} = y) \geq c_{12}^{\widetilde{n}}\,. \tag{4.7}$$

By the Markov property,

$$\begin{aligned}
&P_\omega^y(T_{\overleftarrow{y}} = \infty, T_{y_i} = \infty) \\
= &\sum_{j\neq i} \omega(y, y_j)\beta(y_j) + \left(\sum_{j\neq i} \omega(y, y_j)(1-\beta(y_j))\right) P_\omega^y(T_{\overleftarrow{y}} = \infty, T_{y_i} = \infty).
\end{aligned}$$

This leads to

$$\begin{aligned}
P_\omega^y(T_{\overleftarrow{y}} = \infty, T_{y_i} = \infty) &= \frac{\sum_{j\neq i} A(y_j)\beta(y_j)}{1 + A(y_i) + \sum_{j\neq i} A(y_j)\beta(y_j)} \\
&\geq \frac{1}{\alpha(1+\alpha)} \frac{\sum_{j\neq i}\beta(y_j)}{1 + \sum_{j\neq i}\beta(y_j)} \\
&\geq \frac{1}{2\alpha(1+\alpha)}\left(1 \wedge \sum_{j\neq i}\beta(y_j)\right).
\end{aligned}$$

Similarly, $P_\omega^y(T_{\overleftarrow{y}} = \infty) \geq \frac{1}{2\alpha^2}\left(1 \wedge \sum_{j=1}^{\nu(y)}\beta(y_j)\right)$. Thus, we have for any $|y| = \widetilde{n} - 1$,

$$P_\omega^y(T_{\overleftarrow{y}} = \infty\,,\, T_{\widetilde{u}} = \infty) \geq c_{13}\left(1 \wedge \sum_{y_j \neq \widetilde{u}}\beta(y_j)\right). \tag{4.8}$$

By equations (4.6), (4.7) and (4.8), we have

$$p_1(e,u) \geq c_{13}c_{12}^{\widetilde{n}}\left(1 \wedge \sum_{|x|=\widetilde{n}: x\neq\widetilde{u}}\beta(x)\right).$$

Therefore, arguing over the value of $\widetilde{u}$, we obtain

$$\mathbb{1}_{\{n\geq\widetilde{n}\}}\sum_{|u|=n}\mathbf{E}\left[\frac{1}{[p_1(e,u)]^k}\right] \leq c_{14}\sum_{|y|=\widetilde{n}}\nu(y, n-\tilde{n})\mathbf{E}\left[1 \vee \frac{1}{[\sum_{|x|=\widetilde{n}, x\neq y}\beta(x)]^k}\right],$$

where $c_{14} := (c_{13}c_{12}^n)^{-k}$. By using the Galton–Watson property at generation $\widetilde{n}$,

$$\begin{aligned}
&\sum_{|u|=n} E_{\mathbf{Q}}\left[\frac{\mathbb{1}_{\{u\in\mathbb{T}, Z_n > b_0\}}}{[p_1(e,u)]^k}\,\middle|\,\widetilde{n}\,, Z_0, \ldots, Z_{\widetilde{n}}\right] \\
\leq\; & c_{14}\sum_{|y|=\widetilde{n}} E_{GW}[\nu(y, n-\widetilde{n})] E_{\mathbf{Q}}\left[1 \vee \frac{1}{[\sum_{i=1}^p \beta(i)]^k}\right]_{p=Z_{\widetilde{n}}-1} \\
\leq\; & c_{15} Z_{\widetilde{n}}
\end{aligned}$$

by Lemma 4.2. Integrating over GW completes the proof of (4.5). $\square$

Remark. Lemma 4.3 tells in particular that

$$E_{\mathbf{Q}}\left[\frac{\mathbb{1}_{\{Z_n>b_0\}}}{\beta(e)}\right] \leq E_{\mathbf{Q}}\left[\frac{\mathbb{1}_{\{Z_n>b_0\}}}{P_\omega^e(T_{\overleftarrow{e}}=\infty, T_e^*=\infty)}\right] < \infty \,. \tag{4.9}$$

We deal now with a comparison between RWREs on a tree and one-dimensional RWREs already used in [13]. Let $\mathbb{T}$ be a tree and ω the environment on this tree. Take $x \leq y \in \mathbb{T}$. We look at the path $[\![\overleftarrow{x}, y]\!] = \{\overleftarrow{x} = x_{-1}, x_0, \ldots, x_p = y\}$ defined as the shortest path from $\overleftarrow{x}$ to y, and we consider on it the random walk $(\widetilde{X}_n)$ with probability transitions $\widetilde{\omega}(\overleftarrow{x}, x) = \widetilde{\omega}(y, x_{p-1}) = 1$ and for any $0 \leq i < p$,

$$\begin{aligned}
\widetilde{\omega}(x_i, x_{i+1}) &= \frac{\omega(x_i, x_{i+1})}{\omega(x_i, x_{i+1}) + \omega(x_i, x_i - 1)}\,, \\
\widetilde{\omega}(x_i, x_{i-1}) &= \frac{\omega(x_i, x_{i-1})}{\omega(x_i, x_{i+1}) + \omega(x_i, x_{i-1})}\,.
\end{aligned}$$

Thus we can associate to the pair (x, y) a one-dimensional RWRE on $[\![\overleftarrow{x}, y]\!]$, and we denote by $\widetilde{P}$, $\widetilde{E}$ the probabilities and expectations related to this new RWRE. We observe that under $\mathbb{Q}^x$, the RWRE $(\widetilde{X}_n,\, n \leq T_{\overleftarrow{x}} \wedge T_y)$ has the distribution of the RWRE $(R_n,\, n \leq T_{-1} \wedge T_p)$ introduced in Section 3.2. For any $x, y \in \mathbb{T}$, the event $\{T_x < T_y\}$ means that $T_x < \infty$ and $T_x < T_y$.

Lemma 4.4 *For any $x, y, z \in \mathbb{T}$ with $x \leq z < y$,*

$$\begin{aligned}
P_\omega^z(T_y < T_{\overleftarrow{x}}) &\leq \widetilde{P}_\omega^z(T_y < T_{\overleftarrow{x}})\,, \\
P_\omega^z(T_{\overleftarrow{x}} < T_y) &\leq \widetilde{P}_\omega^z(T_{\overleftarrow{x}} < T_y)\,.
\end{aligned}$$

Proof. Fix $z_1, \ldots z_{n-1}$ in $]\!]\overleftarrow{x}, y[\![$ and $z_n \in [\![\overleftarrow{x}, y]\!]$. Then

$$P_\omega^z(X_1 = z_1, \ldots, X_n = z_n) = \frac{\omega(z, z_1)}{1 - f(z)} \cdots \frac{\omega(z_{n-1}, z_n)}{1 - f(z_{n-1})}$$

where $f(r)$ represents the probability of making an excursion away from the path $[\![\overleftarrow{x}, y]\!]$ from the vertex r. For each $r \in [\![\overleftarrow{x}, y[\![$, call r^+ the child of r which lies in the path. Then $f(r) \leq 1 - \omega(r, r^+) - \omega(r, \overleftarrow{r})$. It follows that

$$\begin{aligned}
P_\omega^z(X_1 = z_1, \ldots, X_n = z_n) &\leq \widetilde{\omega}(z, z_1) \ldots \widetilde{\omega}(z_{n-1}, z_n) \\
&= \widetilde{P}_\omega^z(\widetilde{X}_1 = z_1, \ldots, \widetilde{X}_n = z_n)\,.
\end{aligned}$$

It remains to see that the events $\{T_y < T_x\}$ and $\{T_x < T_y\}$ can be written as an union of disjoint sets of the form $\{X_1 = z_1, \ldots, X_n = z_n\}$. $\square$

The last lemma deals with the one-dimensional RWRE $(R_n)_{n\geq 0}$ defined in Section 3.2.

Lemma 4.5 *For any $n \geq 1$, there exists a number $c_{19}(n)$ such that for any $i > n$ and almost every ω,*

$$E_\omega^0[T_{-1} \wedge T_i] \leq c_{19} E_\omega^n[T_{n-1} \wedge T_i]\,.$$

Proof. Let $i > n \geq 1$. By the Markov property and for $0 < p \leq i$, we have

$$E_\omega^{p-1}[T_{p-2} \wedge T_i] = 1 + \omega(p-1,p)\left\{E_\omega^p[T_{p-1} \wedge T_i] + P_\omega^p(T_{p-1} < T_i) E_\omega^{p-1}[T_{p-2} \wedge T_i]\right\}$$

which gives that $E_\omega^{p-1}[T_{p-2} \wedge T_i] = \frac{1+\omega(p-1,p)E_\omega^p[T_{p-1}\wedge T_i]}{1-\omega(p-1,p)P_\omega^p(T_{p-1}\wedge T_i)}$, so that for some c_{20}, c_{21} and c_{22} we have

$$E_\omega^{p-1}[T_{p-2} \wedge T_i] \leq c_{20} + c_{21} E_\omega^p[T_{p-1} \wedge T_i] \leq c_{22} E_\omega^p[T_{p-1} \wedge T_i].$$

Iterating the inequality over all p from 1 to n gives the desired inequality. $\square$

5 Proof of Theorem 1.3: lower bound

Let $(R_n)_{n\geq 0}$ be the one-dimensional RWRE associated with $\mathbb{T} = \{-1, 0, 1, \ldots\}$ defined in Section 3.2 and $T_i = \inf\{k \geq 0 : R_k = i\}$. Define for any $\lambda \in [0,1]$,

$$m(n,\lambda) := \mathbf{E}\left[\left(E_\omega^0\left[T_{-1} \wedge T_n\right]\right)^\lambda\right]\,, \tag{5.1}$$

and let

$$\lambda_c := \sup\left\{\lambda \geq 0 \,:\, \exists r > q_1 \text{ such that } \sum_{n\geq 0} m(n,\lambda) r^n < \infty\right\}\,. \tag{5.2}$$

We start with a lemma.

Lemma 5.1 *We have $\Lambda \leq \lambda_c$.*

Proof. See Section 8. $\square$

Take a $\lambda \in [0,1]$ such that $\lambda < \Lambda$. By Lemma 5.1, we have $\lambda < \lambda_c$ which in turn implies by (5.2) that there exists an $1 > r > q_1$ such that

$$\sum_{n\geq 0} m(n,\lambda)\,(n+1)r^n < \infty\,. \tag{5.3}$$

Recall the definition of b_0 in Lemma 4.2. Then, by Lemma 4.1, we can define

$$n_0 := \inf\left\{n \geq 1 \,:\, E_{GW}[Z_n \mathbb{1}_{\{Z_n \leq b_0\}}] \leq r^n\right\}\,.$$

Let $\mathbb{T}_{n_0}$ be the subtree of $\mathbb{T}$ defined as follows: y is a child of x in $\mathbb{T}_{n_0}$ if $x \leq y$ and $|y-x| = n_0$. In this new Galton–Watson tree $\mathbb{T}_{n_0}$, we define

$$\mathbb{W} = \mathbb{W}(\mathbb{T}) := \{x \in \mathbb{T}_{n_0} : \forall y \in \mathbb{T}_{n_0}, (y < x) \Rightarrow \nu(y,n_0) \leq b_0\}\,, \tag{5.4}$$

where $\nu(y,n_0)$ is defined in (4.4). We call W_k the size of the k-th generation of $\mathbb{W}$. The subtree $\mathbb{W}$ is a Galton–Watson tree, whose offspring distribution is of mean $E_{GW}[Z_{n_0}\mathbb{1}_{\{Z_{n_0}\leq b_0\}}] \leq r^{n_0}$. In particular, we have for any $k \geq 0$,

$$E_{GW}[W_k] \leq r^{kn_0}\,. \tag{5.5}$$

For any $y \in \mathbb{T}$, we denote by y_{n_0} the youngest ancestor of y belonging to $\mathbb{T}_{n_0}$, or equivalently the unique vertex such that

$$y_{n_0} \leq y, \qquad y_{n_0} \in \mathbb{T}_{n_0}, \qquad \forall\, z \in \mathbb{T}_{n_0}\;\; z \leq y \Rightarrow z \leq y_{n_0}\,.$$

Let

$$\begin{aligned} N_{1,n} &:= \sum_{|y|=n} N(y)\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}\,, \\ N_{2,n} &:= \sum_{|y|=n} N(y)\mathbb{1}_{\{\nu(y_{n_0},n_0)\leq b_0, y_{n_0}\notin\mathbb{W}\}}\,. \end{aligned}$$

Lemma 5.2 *There exists a constant L such that for any $n \geq n_0$:*

$$E_{\mathbb{Q}}[N_{1,n}] \leq L\,, \tag{5.6}$$

$$E_{\mathbb{Q}}[N_{2,n}^{\lambda}] \leq L\,. \tag{5.7}$$

We admit Lemma 5.2 for the time being, and show how it implies Theorem 1.3.

Proof of Theorem 1.3: lower bound. Notice that $\mathbb{W}$ is finite almost surely. Then, there exists a random $K \geq 0$ such that for $n \geq K$, $N_n \leq N_{1,n} + N_{2,n}$. Lemma 5.2 yields that $E_{\mathbb{Q}}[N_n^\lambda \mathbb{1}_{\{n\geq K\}}] \leq L^\lambda + L$ for any $n \geq n_0$. By Fatou's lemma, $\liminf_{n\to\infty} \frac{\sum_{k=K}^n N_k^\lambda}{n} < \infty$. Denote by $(r_k,\, k\geq 0)$ the sequence $(|X_{\Gamma_k}|,\, k \geq 0)$. Notice that for any $k \geq 1$,

$$\Gamma_{k+1} - \Gamma_k = \sum_{i=r_k+1}^{r_{k+1}} N_i\,.$$

It yields that $S(u(n),\lambda) := \sum_{k=1}^{u(n)} (\Gamma_k - \Gamma_{k-1})^\lambda \leq \sum_{i=0}^{r_{u(n)}} N_i^\lambda \leq \sum_{i=0}^n N_i^\lambda$ where, as in Section 3, $u(n)$ is the unique integer such that $\Gamma_{u(n)} \leq \tau_n < \Gamma_{u(n)+1}$. Observe also that $\frac{n}{u(n)}$ tends to $E_{\mathbb{S}}\left[|X_{\Gamma_1}|\right]$. It follows that

$$\liminf_{n\to\infty} \frac{1}{n} S(n,\lambda) \leq \liminf_{n\to\infty} \frac{1}{u(n)} \sum_{k=K}^n N_k^\lambda = E_{\mathbb{S}}\left[|X_{\Gamma_1}|\right] \liminf_{n\to\infty} \frac{1}{n} \sum_{k=K}^n N_k^\lambda < \infty\,.$$

Using equation (3.2) implies that $\limsup_{n\to\infty} \frac{(\Gamma_n)^\lambda}{n} < c_{23}$ for some constant c_{23}. We check that $|X_n| \geq \#\{k : \Gamma_k \leq n\}$ which leads to $|X_n| \geq \frac{n^\lambda}{c_{23}}$ for sufficiently large n. Letting λ go to Λ completes the proof. □

The rest of this section is devoted to the proof of Lemma 5.2. For the sake of clarity, the two estimates, (5.6) and (5.7), are proved in distinct parts.

5.1 Proof of Lemma 5.2: equation (5.6)

For all $y \in \mathbb{T}$, call Y the youngest ancestor of y such that $\nu(Y,n_0) > b_0$. We have

$$E_\omega^e[N(y)] = P_\omega^e(T_y < \infty) E_\omega^y[N(y)] \leq P_\omega^e(T_Y < \infty) E_\omega^y[N(y)]\,.$$

We compute $E_\omega^y[N(y)]$ with a method similar to the one given in [13]. By the Markov property,

$$E_\omega^y[N(y)] = G(y,Y) + P_\omega^y(T_Y < \infty) P_\omega^Y(T_y < \infty) E_\omega^y[N(y)]\,,$$

where $G(y,Y) := E_\omega^y\left[\sum_{k=0}^{T_Y} \mathbb{1}_{\{X_k = y\}}\right]$. When $\nu(y_{n_0},n_0) > b_0$, there exists a constant $c_{24} > 0$ such that $P_\omega^y(T_y^* > T_Y) \geq c_{24}$. Therefore, in this case $G(y,Y) \leq (c_{24})^{-1} =: c_{25}$. It follows that

$$
\begin{aligned}
E_\omega^y[N(y)]\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}} &\le c_{25}\frac{\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}}{1-P_\omega^Y(T_y<\infty)P_\omega^y(T_Y<\infty)}\\
&\le c_{25}\frac{\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}}{1-P_\omega^Y(T_Y^*<\infty)}\\
&\le c_{25}\frac{\mathbb{1}_{\{\nu(y_{n_0},n_0)>b_0\}}}{\gamma(Y)},
\end{aligned}
$$

where $\gamma(x) := P_\omega^x(T_{\overleftarrow{x}}=\infty, T_x^*=\infty)$. Arguing over the value of Y yields that

$$
\begin{aligned}
E_{\mathbb{Q}}[N_{1,n}] &\le c_{25}E_{\mathbf{Q}}\left[\sum_{n-n_0<|z|\le n} P_\omega^e(T_z<\infty)\frac{\mathbb{1}_{\{\nu(z,n_0)>b_0\}}}{\gamma(z)}\right]\\
&= c_{25}E_{\mathbf{Q}}\left[\sum_{n-n_0<|z|\le n} P_\omega^e(T_z<\infty)\right]E_{\mathbf{Q}}\left[\frac{\mathbb{1}_{\{Z_{n_0}>b_0\}}}{\gamma(e)}\right]\\
&\le c_{25}n_0\, c_1\, c_{26}\,,
\end{aligned}
$$

by Lemma 2.1 and equation (4.9). $\square$

5.2 Proof of Lemma 5.2: equation (5.7)

For any $y\in\mathbb{T}$ such that $\nu(y_{n_0},n_0)\le b_0$ and $y_{n_0}\notin\mathbb{W}$, choose $Y_1=Y_1(y)$, $Y_2=Y_2(y)$ and $Y_3=Y_3(y)$, vertices of $\mathbb{T}_{n_0}$, such that

$$
\begin{array}{lll}
Y_1<y, & \nu(Y_1,n_0)>b_0, & \forall\, z\in\mathbb{T}_{n_0},\ Y_1<z\le y\Rightarrow \nu(z,n_0)\le b_0\\
Y_1<Y_2\le y, & & \forall\, z\in\mathbb{T}_{n_0},\ Y_1<z\le y\Rightarrow Y_2\le z\,,\\
y\le Y_3, & \nu(Y_3,n_0)>b_0, & \forall\, z\in\mathbb{T}_{n_0},\ y\le z<Y_3\Rightarrow \nu(z,n_0)\le b_0\,.
\end{array}
$$

By definition, Y_1 is the youngest ancestor of y in $\mathbb{T}_{n_0}$ such that $\nu(Y_1,n_0)>b_0$ and Y_2 the child of Y_1 in $\mathbb{T}_{n_0}$ which is also an ancestor of y. In the rest of the section, $\widetilde{P}_\omega=\widetilde{P}_\omega(Y_1,Y_3)$ and $\widetilde{E}_\omega=\widetilde{E}_\omega(Y_1,Y_3)$ represent the probability and expectation for the one-dimensional RWRE associated to the path $[\![Y_1,Y_3]\!]$, as seen in Lemma 4.4. They depend then on the pair (Y_1,Y_3), which doesn't appear in the notation for sake of brevity. Define for any $n\ge n_0$,

$$
S(n) := E_{\mathbf{Q}}\left[\sum_{|y|=n:Y_1=e}\left[p_1(e,Y_2)^2\beta(Y_3)\right]^{-1}\left(\widetilde{E}_\omega^{Y_2}[T_{\overleftarrow{Y}_2}\wedge T_{Y_3}]\right)^\lambda\right], \tag{5.8}
$$

where $\overleftarrow{Y}_2$ represents as usual the parent of Y_2 in the tree $\mathbb{T}$ and $p_1(u,v)$ is defined in (4.3).

Lemma 5.3 *There exists a constant c_{27} such that for any $n \geq n_0$,*

$$E_{\mathbb{Q}}[N_{2,n}^{\lambda}] \leq c_{27} \sum_{k \geq n_0} S(k)\,.$$

Proof. We observe that

$$E_{\omega}^{e}[N_n^{\lambda}] = E_{\omega}^{e}\left[\left(\sum_{|y|=n} N(y)\right)^{\lambda}\right] \leq E_{\omega}^{e}\left[\sum_{|y|=n} N(y)^{\lambda}\right]$$

since $\lambda \leq 1$. By the Markov property, $E_{\omega}^{e}[\sum_{|y|=n} N(y)^{\lambda}] = \sum_{|y|=n} P_{\omega}^{e}(T_y < \infty) E_{\omega}^{y}[N(y)^{\lambda}]$. An application of Jensen's inequality yields that

$$E_{\omega}^{e}[N_n^{\lambda}] \leq \sum_{|y|=n} P_{\omega}^{e}(T_y < \infty)\left(E_{\omega}^{y}[N(y)]\right)^{\lambda}\,. \tag{5.9}$$

Using the Markov property for any $|y| = n$, we get

$$\begin{aligned} & E_{\omega}^{y}[N(y)] \\ = \;& G(y, Y_1 \wedge Y_3) + E_{\omega}^{y}[N(y)](P_{\omega}^{y}(T_{Y_1} < T_{Y_3}) P_{\omega}^{Y_1}(T_y < \infty) + P_{\omega}^{y}(T_{Y_3} < T_{Y_1}) P_{\omega}^{Y_3}(T_y < \infty))\,, \end{aligned}$$

where $G(y, Y_1 \wedge Y_3) := E_{\omega}^{y}\left[\sum_{k=0}^{T_{Y_1} \wedge T_{Y_3}} \mathbb{1}_{\{X_k = y\}}\right]$. Accordingly,

$$E_{\omega}^{y}[N(y)] = \frac{G(y, Y_1 \wedge Y_3)}{1 - P_{\omega}^{y}(T_{Y_1} < T_{Y_3}) P_{\omega}^{Y_1}(T_y < \infty) - P_{\omega}^{y}(T_{Y_3} < T_{Y_1}) P_{\omega}^{Y_3}(T_y < \infty)}\,.$$

Notice that $\left[1 - P_{\omega}^{y}(T_{Y_1} < T_{Y_3}) P_{\omega}^{Y_1}(T_y < \infty) - P_{\omega}^{y}(T_{Y_3} < T_{Y_1}) P_{\omega}^{Y_3}(T_y < \infty)\right]^{-1}$ is the expected number of times when the walk go from y to Y_1 or Y_3 and then returns to y, which is naturally smaller than $E_{\omega}^{y}[N(Y_1) + N(Y_3)]$. We have

$$\begin{aligned} E_{\omega}^{y}[N(Y_1)] &= P_{\omega}^{y}(T_{Y_1} < \infty)\left[1 - P_{\omega}^{Y_1}(T_{Y_1}^{*} < \infty)\right]^{-1} \\ &\leq [p_1(Y_1, Y_2)]^{-1}\,, \end{aligned}$$

where as before $p_1(Y_1, Y_2) = P_{\omega}^{Y_1}\left(T_{\overleftarrow{Y_1}} = \infty\,,\, T_{Y_1}^{*} = \infty\,,\, T_{Y_2} = \infty\right)$. Similarly $E_{\omega}^{y}[N(Y_3)] \leq [\beta(Y_3)]^{-1}$. We obtain

$$P_{\omega}^{e}(T_y < \infty)\left(E_{\omega}^{y}[N(y)]\right)^{\lambda} \leq [p_1(Y_1, Y_2)\beta(Y_3)]^{-1} P_{\omega}^{e}(T_y < \infty)\left(G(y, Y_1 \wedge Y_3)\right)^{\lambda}\,. \tag{5.10}$$

We deduce from the Markov property that $P_{\omega}^{e}(T_y < \infty) = P_{\omega}^{e}(T_{Y_1} < \infty) P_{\omega}^{Y_1}(T_y < \infty)$ and $P_{\omega}^{Y_1}(T_y < \infty) = G(Y_1, y) P_{\omega}^{Y_1}(T_y < T_{Y_1}^{*})$ where $G(Y_1, y) := E_{\omega}^{Y_1}\left[\sum_{k=0}^{T_y} \mathbb{1}_{\{X_k = Y_1\}}\right]$. By Lemma

4.4, we have $P_\omega^{Y_1}(T_y < T_{\overleftarrow{Y}_1}) \le \widetilde{P}_\omega^{Y_1}(T_y < T_{\overleftarrow{Y}_1})$. In words, it means that the probability to escape by y is lower for the RWRE on the tree than for the restriction of the walk on $[\![Y_1, y]\!]$. Furthermore $G(Y_1, y) \le E_\omega^{Y_1}[N(Y_1)] \le [p_1(Y_1, Y_2)]^{-1}$, so that

$$
\begin{aligned}
P_\omega^e(T_y < \infty) &\le P_\omega^e(T_{Y_1} < \infty)\widetilde{P}_\omega^{Y_1}(T_y < T_{\overleftarrow{Y}_1})[p_1(Y_1, Y_2)]^{-1} \\
&\le P_\omega^e(T_{Y_1} < \infty)\left(\widetilde{P}_\omega^{Y_1}(T_y < T_{\overleftarrow{Y}_1})\right)^\lambda [p_1(Y_1, Y_2)]^{-1}. \qquad (5.11)
\end{aligned}
$$

We observe that

$$
G(y, Y_1 \wedge Y_3) = \left[1 - P_\omega^y(T_y^* < T_{Y_1} \wedge T_{Y_3})\right]^{-1}. \qquad (5.12)
$$

Call y_3 the unique child of y such that $y_3 \le Y_3$. Consequently,

$$
\begin{aligned}
&P_\omega^y(T_y^* < T_{Y_1} \wedge T_{Y_3}) \\
\le\ & [1 - \omega(y, y_3) - \omega(y, \overleftarrow{y})] + \omega(y, \overleftarrow{y})P_\omega^{\overleftarrow{y}}(T_y < T_{Y_1}) + \omega(y, y_3)P_\omega^{y_3}(T_y < T_{Y_3}).
\end{aligned}
$$

By Lemma 4.4, we have

$$
\begin{aligned}
P_\omega^{\overleftarrow{y}}(T_y < T_{Y_1}) &\le \widetilde{P}_\omega^{\overleftarrow{y}}(T_y < T_{Y_1}), \\
P_\omega^{y_3}(T_y < T_{Y_3}) &\le \widetilde{P}_\omega^{y_3}(T_y < T_{Y_3}).
\end{aligned}
$$

Equation (5.12) becomes $G(y, Y_1 \wedge Y_3) \le (\omega(y, y_3) + \omega(y, \overleftarrow{y}))^{-1}\widetilde{G}(y, Y_1 \wedge Y_3)$ where $\widetilde{G}(y, Y_1 \wedge Y_3)$ stands for the expectation of the number of times the one-dimensional RWRE associated to the pair (Y_1, Y_3) by Lemma 4.4 crosses y before reaching Y_1 or Y_3 when started from y. Since $\nu(y) \le b_0$, there exists a constant c_{28} such that $(\omega(y, \overleftarrow{y}) + \omega(y, y_3))^{-1} \le c_{28}$. It yields

$$
G(y, Y_1 \wedge Y_3) \le c_{28}\, \widetilde{G}(y, Y_1 \wedge Y_3). \qquad (5.13)
$$

Finally, using (5.11), (5.13), and the following inequality,

$$
\widetilde{P}_\omega^{Y_1}(T_y < T_{\overleftarrow{Y}_1})\, \widetilde{G}(y, Y_1 \wedge Y_3) \le \widetilde{E}_\omega^{Y_1}[T_{\overleftarrow{Y}_1} \wedge T_{Y_3}],
$$

we get

$$
P_\omega^e(T_y < \infty)\, (G(y, Y_1 \wedge Y_3))^\lambda \le \frac{c_{28}}{p_1(Y_1, Y_2)} P_\omega^e(T_{Y_1} < \infty)(\widetilde{E}_\omega^{Y_1}[T_{\overleftarrow{Y}_1} \wedge T_{Y_3}])^\lambda.
$$

By Lemma 4.5, for any $y \in \mathbb{T}$, we have

$$
\widetilde{E}_\omega^{Y_1}[T_{\overleftarrow{Y}_1} \wedge T_{Y_3}] \le c_{19}(n_0)\widetilde{E}_\omega^{Y_2}[T_{\overleftarrow{Y}_2} \wedge T_{Y_3}].
$$

It follows that

$$(5.14)\quad P^e_\omega(T_y<\infty)\,(G(y,Y_1\wedge Y_3))^\lambda \le \frac{c_{28}c_{19}^\lambda}{p_1(Y_1,Y_2)} P^e_\omega(T_{Y_1}<\infty)(\widetilde{E}^{Y_2}_\omega[T_{\overleftarrow{Y}_2}\wedge T_{Y_3}])^\lambda\,.$$

In view of equations (5.10) and (5.14), we obtain

$$P^e_\omega(T_y<\infty)\,(E^y_\omega[N(y)])^\lambda \;\;\le\;\; c_{29}\,P^e_\omega(T_{Y_1}<\infty)H(Y_1,y,Y_3)$$

where

$$H(Y_1,y,Y_3):=\left[p_1(Y_1,Y_2)^2\beta(Y_3)\right]^{-1}\left(\widetilde{E}^{Y_2}_\omega[T_{\overleftarrow{Y}_2}\wedge T_{Y_3}]\right)^\lambda\,.$$

By equation (5.9), it implies that

$$E_{\mathbb{Q}}[N^\lambda_{2,n}]\le c_{29}\,E_{\mathbf{Q}}\left[\sum_{|y|=n}P^e_\omega(T_{Y_1}<\infty)H(Y_1,y,Y_3)\right]\,.$$

Arguing over the value of Y_1 gives

$$\begin{aligned}
E_{\mathbb{Q}}[N^\lambda_{2,n}] &\le c_{29}\,E_{\mathbf{Q}}\left[\sum_{|z|\le n-n_0}P^e_\omega(T_z<\infty)\left(\sum_{|y|=n,Y_1=z}H(z,y,Y_3)\right)\right]\\
&= c_{29}\,E_{\mathbf{Q}}\left[\sum_{|z|\le n-n_0}P^e_\omega(T_z<\infty)E_{\mathbf{Q}}\left[\sum_{|y|=n-|z|,Y_1=e}H(e,y,Y_3)\right]\right]\\
&= c_{29}\,E_{\mathbf{Q}}\left[\sum_{|z|\le n-n_0}P^e_\omega(T_z<\infty)S(n-|z|)\right]\,,
\end{aligned}$$

by equation (5.8). Lemma 2.1 yields that

$$\begin{aligned}
E_{\mathbb{Q}}[N^\lambda_{2,n}] &\le c_1c_{29}\sum_{k=n_0}^{n}S(k)\\
&\le c_1c_{29}\sum_{k\ge n_0}S(k)\,.\;\square
\end{aligned}$$

We call as before $m(n,\lambda):=\mathbf{E}\left[\left(E^0_\omega\left[T_{-1}\wedge T_n\right]\right)^\lambda\right]$ for the one-dimensional RWRE $(R_n)_{n\ge0}$. The following lemma gives an estimate of $S(n)$.

Lemma 5.4 *There exists a constant c_{30} such that for any $\ell\ge 0$,*

$$S(\ell+n_0)\le c_{30}\sum_{i\ge\ell}m(i,\lambda)r^i\,.$$

Proof. Let $\ell \geq 0$ and $f(Y_2, Y_3) := \left(\widetilde{E}^{Y_2}[T_{\overleftarrow{Y}_2} \wedge T_{Y_3}]\right)^{\lambda}$. We have

$$\begin{aligned} S(\ell + n_0) &= E_{\mathbf{Q}}\left[\sum_{|y|=\ell+n_0: Y_1=e} \left[p_1(e, Y_2)^2 \beta(Y_3)\right]^{-1} f(Y_2, Y_3)\right] \\ &= E_{\mathbf{Q}}\left[\sum_{|u|=n_0} [p_1(e, u)]^{-2} \sum_{|y|=\ell+n_0: Y_2=u} f(u, Y_3)\, [\beta(Y_3)]^{-1}\right]. \end{aligned}$$

If we call $\mathbb{T}_u$ the subtree of $\mathbb{T}$ rooted in u, we observe that

$$\sum_{|y|=\ell+n_0: Y_2=u} f(u, Y_3)\, [\beta(Y_3)]^{-1} \leq 1\!\!1_{\{Z_{n_0} > b_0\}} \sum_{|z| \geq \ell+n_0: z \in \mathbb{W}(\mathbb{T}_u)} f(u, z)\, [\beta(z)]^{-1}\, 1\!\!1_{\{\nu(z, n_0) > b_0\}},$$

where $\mathbb{W}$ was defined in equation (5.4). The Galton–Watson property yields that

$$\begin{aligned} S(\ell + n_0) &\leq E_{\mathbf{Q}}\left[\sum_{|u|=n_0} \frac{1\!\!1_{\{Z_{n_0} > b_0\}}}{p_1(e, u)^2}\right] E_{\mathbf{Q}}\left[\sum_{|z| \geq \ell, z \in \mathbb{W}} f(e, z)\, [\beta(z)]^{-1}\, 1\!\!1_{\{\nu(z, n_0) > b_0\}}\right] \\ &= E_{\mathbf{Q}}\left[\sum_{|u|=n_0} \frac{1\!\!1_{\{Z_{n_0} > b_0\}}}{p_1(e, u)^2}\right] E_{\mathbf{Q}}\left[\sum_{|z| \geq \ell, z \in \mathbb{W}} f(e, z)\right] E_{\mathbf{Q}}\left[\frac{1\!\!1_{\{Z_{n_0} > b_0\}}}{\beta(e)}\right] \\ &\leq c_{31} E_{\mathbf{Q}}\left[\sum_{|z| \geq \ell, z \in \mathbb{W}} f(e, z)\right], \end{aligned}$$

by Lemma 4.3 and equation (4.9). The proof follows then from

$$\begin{aligned} E_{\mathbf{Q}}\left[\sum_{|z| \geq \ell, z \in \mathbb{W}} f(e, z)\right] &= E_{GW}\left[\sum_{|z| \geq \ell, z \in \mathbb{W}} m(|z|, \lambda)\right] \\ &= \sum_{i: in_0 \geq \ell} m(in_0, \lambda) E_{GW}[W_i] \leq \sum_{in_0 \geq \ell} m(in_0, \lambda) r^{in_0}, \end{aligned}$$

where the last inequality comes from equation (5.5). $\square$

We are now able to prove (5.7).

Proof of Lemma 5.2, equation (5.7). By Lemma 5.3,

$$E_{\mathbb{Q}}[N_{2,n}^{\lambda}] \leq c_{27} \sum_{\ell \geq 0} S(\ell + n_0).$$

Lemma 5.4 tells that

$$\begin{aligned}\sum_{\ell\geq 0} S(\ell+n_0) &\leq c_{30}\sum_{i\geq\ell\geq 0} m(i,\lambda)r^i \\ &= c_{30}\sum_{i\geq 0}(i+1)m(i,\lambda)r^i\,,\end{aligned}$$

which is finite by equation (5.3). □

6 Proof of Theorem 1.1

If we suppose that $\Lambda < 1$, then Theorem 1.3 ensures that $\frac{|X_n|}{n}$ tends to 0. Suppose now that $\Lambda > 1$. Take $\lambda = 1$ in the proof of the lower bound of Theorem 1.3 in Section 5 to see that $|X_n| \geq \frac{n}{c_{23}}$ for sufficiently large n, which proves the positivity of the speed in this case. Theorem 1.1 is proved. □

7 Proof of Theorem 1.4

When $b \geq 3$, Theorem 1.4 follows immediately from Theorem 1.5. In the rest of this section, we assume that $\mathbb{T}$ is a binary tree. Thanks to the correspondence between RWRE and LERRW mentioned in the introduction, we only have to prove the positivity of the speed for a RWRE on the binary tree such that the density of $\omega(y,z)$ on $(0,1)$ is given by

$$f_0(x) = 1 \quad \text{if } z = \overleftarrow{y} \tag{7.1}$$

$$f_1(x) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}x^{-1/2}(1-x)^{1/2} \quad \text{if } z \text{ is a child of } y. \tag{7.2}$$

We propose to prove three lemmas before handling the proof of the theorem.

Lemma 7.1 *We have for any* $0 < \delta < 1$,

$$\mathbf{E}\left[\frac{1}{\beta^\delta}\right] < \infty\,.$$

Proof. By equation (2.1), for any $y \in \mathbb{T}$,

$$\begin{aligned}\frac{1}{\beta(y)^\delta} &\leq \left(1+\min_{i=1,2}\frac{1}{A(y_i)\beta(y_i)}\right)^\delta \\ &\leq 1+\min_{i=1,2}\frac{1}{A(y_i)^\delta\beta(y_i)^\delta}\,.\end{aligned}$$

Notice that by (7.1),

$$\mathbf{E}\left[\min_{i=1,2}\frac{1}{A(y_i)^\delta}\right] \leq 2^\delta\,\mathbf{E}\left[\left(\frac{1}{A(y_1)+A(y_2)}\right)^\delta\right] = 2^\delta\,\mathbf{E}\left[\left(\frac{\omega(y,\overleftarrow{y})}{1-\omega(y,\overleftarrow{y})}\right)^\delta\right] < \infty\,.$$

The proof is therefore the proof of Lemma 2.2 when replacing $A(y)$ and $\beta(y)$ respectively by $A(y)^\delta$ and $\beta(y)^\delta$. $\square$

Recall that for any $y \in \mathbb{T}$, $\gamma(y) := P^y_\omega(T_{\overleftarrow{y}} = \infty,\, T^*_y = \infty)$.

Lemma 7.2 *There exists $\mu \in (0,1)$ such that for any $\varepsilon \in (0,1)$, we have*

$$\mathbf{E}\left[\left(\frac{1\!\!1_{\{\omega(e,\overleftarrow{e})\leq 1-\varepsilon\}}}{\gamma(e)}\right)^{1/\mu}\right] < \infty\,.$$

Proof. We see that

$$\frac{1}{\gamma(e)} = \frac{1}{\omega(e,e_1)\beta(e_1)+\omega(e,e_2)\beta(e_2)} \leq \min_{i=1,2}\frac{1}{\omega(e,e_i)\beta(e_i)}\,.$$

Let $\mu \in (0,1)$ and $\varepsilon \in (0,1)$. We compute $\mathbf{P}(\omega(e,\overleftarrow{e}) \leq 1-\varepsilon\,,\, \min_{i=1,2}\left\{[\omega(e,e_i)\beta(e_i)]^{-1/\mu}\right\} > n)$ for $n \in \mathbb{R}^*_+$. We observe that $\{\omega(e,\overleftarrow{e}) \leq 1-\varepsilon\} \subset \{\omega(e,e_1) \geq \varepsilon/2\} \cup \{\omega(e,e_2) \geq \varepsilon/2\}$. By symmetry,

$$\begin{aligned}
&\mathbf{P}\left(\omega(e,\overleftarrow{e}) \leq 1-\varepsilon\,,\, \min_{i=1,2}\left\{[\omega(e,e_i)\beta(e_i)]^{-1/\mu}\right\} > n\right)\\
\leq\;& 2\mathbf{P}\left(\omega(e,e_2) \geq \varepsilon/2\,,\, \min_{i=1,2}\left\{[\omega(e,e_i)\beta(e_i)]^{-1/\mu}\right\} > n\right)\\
\leq\;& 2\mathbf{P}\left(\beta(e_2)^{-1} > n^\mu\varepsilon/2\,,\, [\omega(e,e_1)\beta(e_1)]^{-1/\mu} > n\right)\\
\leq\;& 2\mathbf{P}\left(\beta(e_2)^{-1} > n^\mu\varepsilon/2\,,\, \omega(e,e_1) \leq n^{-1/2}\right) + 2\mathbf{P}\left(\beta(e_2)^{-1} > n^\mu\varepsilon/2\,,\, \beta(e_1)^{-1} > n^{\mu-1/2}\right)\\
=:\;& 2\mathbf{P}(E_1) + 2\mathbf{P}(E_2)\,.
\end{aligned}$$

Let $0 < \delta < 1$. We have by (7.2) and Lemma 7.1,

$$\begin{aligned}
\mathbf{P}(E_1) &= \mathbf{P}\left(\omega(e,e_1) \leq n^{-1/2}\right)\mathbf{P}\left(\beta(e_2)^{-1} > n^\mu\varepsilon/2\right)\\
&\leq c_{32} n^{-1/4} n^{-\delta\mu}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{P}(E_2) &= \mathbf{P}\left(\beta(e_1)^{-1} > n^{\mu-1/2}\right)\mathbf{P}\left(\beta(e_2)^{-1} > n^\mu\varepsilon/2\right)\\
&\leq c_{33} n^{-\delta(\mu-1/2)} n^{-\delta\mu}\,.
\end{aligned}$$

It suffices to take $1/4+\delta\mu>1$ and $\delta(2\mu-1/2)>1$ to complete the proof, for example by taking $\delta=4/5$ and $\mu=19/20$. $\square$

Let $\varepsilon\in(0,1/3)$ be such that

$$\mathbf{E}\left[(\#\{i\,:\,\omega(e_i,e)>1-\varepsilon\})^{\frac{2-\mu}{1-\mu}}\right]<1\,. \tag{7.3}$$

Denote by $\mathbb{U}$ the set of the root and all the vertices y such that for any vertex $x\in\mathbb{T}$ with $e<x\leq y$, we have $\omega(x,\overleftarrow{x})>1-\varepsilon$; we observe that by (7.3), $\mathbb{U}$ is a subcritical Galton–Watson tree. Denote by U_k the size of the generation k.

Lemma 7.3 *There exists a constant $c_{34}<1$ such that for any $k\geq 0$*

$$\mathbf{E}\left[U_k^{1/(1-\mu)}\right]\leq c_{34}^k\,.$$

Proof. By Galton–Watson property,

$$\mathbf{E}\left[U_{k+1}^{1/(1-\mu)}\right]=\mathbf{E}\left[\left(\sum_{i=1}^{U_1}U_k^{(i)}\right)^{1/(1-\mu)}\right]$$

where conditionally on U_1, $U_k^{(i)}$, $i\geq 1$ is a family of i.i.d random variables distributed as U_k. Since $(\sum_{i=1}^n a_i)^p\leq n^p\sum_{i=1}^p a_i^p$ (for $p>0$ and $a_i\geq 0$), it yields that

$$\begin{aligned}\mathbf{E}\left[U_{k+1}^{1/(1-\mu)}\right] &\leq \mathbf{E}\left[U_1^{1/(1-\mu)}\sum_{i=1}^{U_1}\left(U_k^{(i)}\right)^{1/(1-\mu)}\right]\\ &= \mathbf{E}\left[U_1^{\frac{2-\mu}{1-\mu}}\right]\mathbf{E}\left[U_k^{1/(1-\mu)}\right]\,.\end{aligned}$$

The proof follows from equation (7.3). $\square$

We are now able to complete the proof of Theorem 1.4.

Proof of Theorem 1.4 : the binary tree case. We suppose without loss of generality that $\omega(e,\overleftarrow{e})\leq 1-\varepsilon$. For any vertex y, we call Y the youngest ancestor of y such that $\omega(Y,\overleftarrow{Y})\leq 1-\varepsilon$. We have for any $n\geq 0$,

$$E_\omega^e[N_n]=\sum_{|y|=n}P_\omega^e(T_y<\infty)E_\omega^y[N(y)]\,,$$

where, as before, $N(y) := \sum_{k\geq 0} \mathbb{1}_{\{X_k=y\}}$ and $N_n = \sum_{|y|=n} N(y)$. By the Markov property,

$$E_\omega^y[N(y)] \;=\; G(y,Y) + P_\omega^y(T_Y<\infty)P_\omega^Y(T_y<\infty)E_\omega^y[N(y)]\,,$$

where $G(y,Y) := E_\omega^y\left[\sum_{k=0}^{T_Y} \mathbb{1}_{\{X_k=y\}}\right]$. It yields that

$$\begin{aligned}
E_\omega^e[N_n] &= \sum_{|y|=n} P_\omega^e(T_y<\infty)\frac{G(y,Y)}{1-P_\omega^Y(T_y<\infty)P_\omega^y(T_Y<\infty)} \\
&\leq \sum_{|y|=n} P_\omega^e(T_y<\infty)\frac{G(y,Y)}{1-P_\omega^Y(T_Y^*<\infty)} \\
&\leq \sum_{|y|=n} P_\omega^e(T_y<\infty)\frac{G(y,Y)}{\gamma(Y)}\,.
\end{aligned}$$

By coupling the walk on $[\![y,Y]\!]$ with a one-dimensional random walk, we see that $P_\omega^y(T_y^*< T_Y)\leq \varepsilon+(1-\varepsilon)\frac{\varepsilon}{1-\varepsilon} = 2\varepsilon\leq 2/3$, so that $G(y,Y)\leq 3$. On the other hand, $P_\omega^e(T_y<\infty)\leq P_\omega^e(T_Y<\infty)$. Therefore,

$$\begin{aligned}
\mathbb{E}[N_n] &\leq 3\mathbf{E}\left[\sum_{|y|=n} P_\omega^e(T_Y<\infty)\frac{1}{\gamma(Y)}\right] \\
&= 3\mathbf{E}\left[\sum_{|y|=n}\sum_{z=Y} P_\omega^e(T_z<\infty)\frac{1}{\gamma(z)}\right] \\
&= 3\mathbf{E}\left[\sum_{|z|\leq n} P_\omega^e(T_z<\infty)\sum_{|y|=n:Y=z}\frac{1}{\gamma(z)}\right]\,.
\end{aligned}$$

By independence and stationarity of the environment,

$$\begin{aligned}
\mathbb{E}[N_n] &\leq 3\sum_{|z|\leq n}\mathbb{P}(T_z<\infty)\mathbf{E}\left[\sum_{|y|=n-|z|:Y=e}\frac{1}{\gamma(e)}\right] \\
&= 3\sum_{|z|\leq n}\mathbb{P}(T_z<\infty)\mathbf{E}\left[\frac{\mathbb{1}_{\{\omega(e,\overleftarrow{e})\leq 1-\varepsilon\}}U_{n-|z|}}{\gamma(e)}\right] \\
&\leq 3\sum_{|z|\leq n}\mathbb{P}(T_z<\infty)\mathbf{E}\left[\left(\frac{\mathbb{1}_{\omega(e,\overleftarrow{e})\leq 1-\varepsilon}}{\gamma(e)}\right)^{1/\mu}\right]^{\mu}\mathbf{E}\left[U_{n-|z|}^{1/(1-\mu)}\right]^{1-\mu}\,,
\end{aligned}$$

by the Hölder inequality. We use Lemmas 7.2 and 7.3 to see that

$$\mathbb{E}[N_n]\leq c_{35}\sum_{|z|\leq n}\mathbb{P}(T(z)<\infty)c_{36}^{n-|z|}.$$

By Lemma 2.1,

$$\mathbb{E}[N_n] \le c_{35}c_1 \sum_{k=0}^{n} c_{36}^k < c_{35}c_1/(1-c_{36})\,.$$

Since $\tau_n \le \sum_{k=-1}^{n} N_k$ and $N_{-1} \le N_0$, where $\tau_n := \inf\{k \ge 0 : |X_k| = n\}$ as before, we have $\mathbb{E}[\tau_n] \le c_{37}\, n$. Fatou's lemma yields that $\mathbb{P}$-almost surely, $\liminf_{n\to\infty} \frac{\tau_n}{n} < \infty$, which proves that $v > 0$ in view of the relation $\lim_{n\to\infty} \frac{\tau_n}{n} = \frac{1}{v}$. $\square$

8 Proof of Lemmas 5.1 and 3.2

We consider the one-dimensional RWRE $(R_n)_{n\ge 0}$ when we consider the case $\mathbb{T} = \{-1, 0, 1, \ldots\}$. This RWRE is such that the random variables $A(i)$, $i \ge 0$ are independent and have the distribution of A, when we set for $i \ge 0$,

$$A(i) := \frac{\omega(i, i+1)}{\omega(i, i-1)}$$

with $\omega(y, z)$ the quenched probability to jump from y to z. We recall that, as defined in equations (3.3) and (5.1),

$$\begin{aligned} p\,(n, a) &:= \mathbb{P}^0(T_{-1} \wedge T_n > a)\,, \\ m(n, \lambda) &:= \mathbf{E}\left[\left(E_\omega^0\left[T_{-1} \wedge T_n\right]\right)^\lambda\right]\,. \end{aligned}$$

We study the walk $(R_n)_{n\ge 0}$ through its potential. We introduce for $p \ge i \ge 0$, $V(0) = 0$ and

$$\begin{aligned} V(i) &= -\sum_{k=0}^{i-1} \ln(A(k))\,, \\ M(i) &= \max_{0\le k\le i} V(k)\,, \\ H_1(i) &= \max_{0\le k\le i} V(k) - V(i)\,, \\ H_2(i,p) &= \max_{i\le k\le p} V(k) - V(i)\,. \end{aligned}$$

Let us introduce for $t \in \mathbb{R}$ the Laplace transform $\mathbf{E}[A^t]$, and define $\phi(t) := \ln(\mathbf{E}[A^t])$. Denote by I its Legendre transform $I(x) = \sup\{tx - \phi(t), t \in \mathbb{R}\}$ where $x \in \mathbb{R}$. Let also

$$[a, b] := [\text{ess inf}(\ln A), \text{ess sup}(\ln A)]\,.$$

Two situations occur. If $a = b$, it means that A is a constant almost surely. In this case, $I(x) = 0$ if $x = a$ and is infinite otherwise. If $a < b$, then I is finite on $]a, b[$ and infinite on

$\mathbb{R}\backslash[a,b]$. Moreover, for any $x \in]a,b[$, we have $I'(x) = t(x)$ where $t(x)$ is the real such that $I(x) = xt(x) - \phi(t(x))$, or, equivalently, $x = \phi'(t(x))$.

We define and compute two useful parameters. Call $\mathcal{D} := \{x_1,\, x_2,\, ,z_1,\, z_2 \in \mathbb{R}_+^4, z_1 + z_2 \leq 1\}$. Define for $0 < \lambda \leq 1$, and with the convention that $0 \times \infty := 0$,

$$
\begin{aligned}
L(\lambda) &:= \sup_{\mathcal{D}} \left\{ \Big((x_1 z_1) \wedge (x_2 z_2) \Big) \lambda - I(-x_1) z_1 - I(x_2) z_2 \right\}, && (8.1)\\
L' &:= \sup \left\{ \frac{x_1 + x_2}{x_1 x_2} \ln(q_1) - \frac{I(-x_1)}{x_1} - \frac{I(x_2)}{x_2}\,,\, x_1, x_2 > 0 \right\}. && (8.2)
\end{aligned}
$$

If $q_1 = 0$, we set $L' = -\infty$. Notice that $L(\lambda) \geq 0$ is necessarily reached for $x_1 z_1 = x_2 z_2$. It yields that

$$
L(\lambda) = 0 \vee \sup \left\{ \frac{x_1 x_2}{x_1 + x_2} \lambda - I(-x_1) \frac{x_2}{x_1 + x_2} - I(x_2) \frac{x_1}{x_1 + x_2}\,,\, x_1, x_2 > 0 \right\}, \tag{8.3}
$$

where $c \vee d := \max(c,d)$. The computation of $L(\lambda)$ and L' is done in the following lemma.

Lemma 8.1 *We have*

$$
\begin{aligned}
L(\lambda) &= 0 \vee \phi(\bar{t}\,), && (8.4)\\
L' &= -\Lambda\,, && (8.5)
\end{aligned}
$$

where $\bar{t}$ verifies $\phi(\bar{t}\,) = \phi(\bar{t} + \lambda)$ if it exists and $\bar{t} := 0$ otherwise.

Proof. When A is a constant almost surely, $L(\lambda) = 0$ and (8.4) is true. Therefore we assume that $a < b$. Considering equation (8.3), we see that if $L(\lambda) > 0$, then $L(\lambda)$ is reached by a pair (x_1, x_2) which satisfies:

$$
\begin{aligned}
\lambda \frac{x_2}{x_1 + x_2} + \frac{I(-x_1)}{x_1 + x_2} + I'(-x_1) - \frac{I(x_2)}{x_1 + x_2} &= 0\,, && (8.6)\\
\lambda \frac{x_1}{x_1 + x_2} - \frac{I(-x_1)}{x_1 + x_2} + \frac{I(x_2)}{x_1 + x_2} - I'(x_2) &= 0\,. && (8.7)
\end{aligned}
$$

We deduce from equations (8.6) and (8.7) that $I'(x_2) - I'(-x_1) = \lambda$, i.e. $t(x_2) - t(-x_1) = \lambda$. Plugging this into (8.3) yields

$$
L(\lambda) = 0 \vee \sup \left\{ \frac{\phi(t)\phi'(t+\lambda) - \phi(t+\lambda)\phi'(t)}{\phi'(t+\lambda) - \phi'(t)}, t \in \mathbb{R}, \phi'(t) < 0, \phi'(t+\lambda) > 0 \right\}.
$$

Let $h(t) := \frac{\phi(t)\phi'(t+\lambda) - \phi(t+\lambda)\phi'(t)}{\phi'(t+\lambda) - \phi'(t)}$. Then $L(\lambda) = 0 \vee h(\bar{t}\,)$ where $\bar{t}$ verifies $h'(\bar{t}\,) = 0$, which is equivalent to say that $\phi(\bar{t}\,) = \phi(\bar{t} + \lambda)$. We find that $h(\bar{t}\,) = \phi(\bar{t}\,)$, which gives (8.4). The computation of (8.5) is similar and is therefore omitted. $\square$

8.1 Proof of Lemma 5.1

We begin by some notation. Let $A > 0$ and $B > 0$ be two expressions which can depend on any variable, and in particular on n. We say that $A \lesssim B$ if we can find a function f of the variable n such that $\lim_{n\to\infty} \frac{1}{n}\ln(f(n)) = 0$ and $A \leq f(n)B$. We say that $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. By circuit analogy (see [5]), we find for $0 \leq i \leq n$,

$$P_\omega^0\left(T_i < T_{-1}\right) = \frac{1}{e^{V(0)} + e^{V(1)} + \ldots + e^{V(i)}}\,.$$

It follows that

$$\frac{e^{-M(i)}}{n+1} \leq P_\omega^0\left(T_i < T_{-1}\right) \leq e^{-M(i)}\,. \tag{8.8}$$

We deduce also that

$$\frac{e^{-H_2(i,n)}}{n+1} \leq P_\omega^{i+1}\left(T_n < T_i\right) \leq e^{-H_2(i,n)}\,, \tag{8.9}$$

$$\frac{e^{-H_1(i)}}{n+1} \leq P_\omega^{i-1}\left(T_{-1} < T_i\right) \leq e^{-H_1(i)}\,. \tag{8.10}$$

Finally, the quenched expectation $G\left(i, -1\wedge n\right)$ of the number of times the walk starting from i returns to i before reaching -1 or n verifies

$$G\left(i, -1\wedge n\right) \;=\; \left\{\omega(i,i-1)P_\omega^{i-1}\left(T_{-1} < T_i\right) + \omega(i,i+1)P_\omega^{i+1}\left(T_n < T_i\right)\right\}^{-1}\,,$$

so that

$$c_{37}e^{H_1(i)\wedge H_2(i,n)} \leq G(i,-1\wedge n) \leq c_{38}(n+1)e^{H_1(i)\wedge H_2(i,n)}\,.$$

Since $E_\omega^0[T_{-1}\wedge T_n] = 1 + \sum_{i=0}^{n-1} P_\omega^0\left(T_i < T_{-1}\right) G\left(i,-1\wedge n\right)$, we get

$$1 + \frac{c_{37}}{n+1}\max_{0\leq i\leq n} e^{-M(i)+H_1(i)\wedge H_2(i,n)} \leq E_\omega^0[T_{-1}\wedge T_n] \leq 1 + c_{38}n(n+1)\max_{0\leq i\leq n} e^{-M(i)+H_1(i)\wedge H_2(i,n)}\,.$$

As a result,

$$\mathbf{E}[\left(E_\omega^0[T_{-1}\wedge T_n]\right)^\lambda] \simeq \max_{0\leq i\leq n}\mathbf{E}\left[e^{\lambda[-M(i)+H_1(i)\wedge H_2(i,n)]}\right]\,. \tag{8.11}$$

We proceed to the proof of Lemma 5.1. Let $\eta > 0$ and $0 \leq i \leq n$. Let $\varepsilon > 0$ be such that $(|a|\vee|b|)\varepsilon < \eta$. For fixed i and n, we denote by K_1 and K_2 the integers such that

$$\begin{aligned} K_1\eta \leq \quad & H_1(i) & < (K_1+1)\eta\,,\\ K_2\eta \leq \quad & H_2(i,n) & < (K_2+1)\eta\,. \end{aligned}$$

Similarly, let L_1 and L_2 be integers such that

$$\begin{array}{lll} \exists \;\; L_1\lfloor \varepsilon n\rfloor \leq x < (L_1+1)\lfloor \varepsilon n\rfloor & \text{such that} & H_1(i) = V(i-x) - V(i)\,, \\ \exists \;\; L_2\lfloor \varepsilon n\rfloor \leq y < (L_2+1)\lfloor \varepsilon n\rfloor & \text{such that} & H_2(i,n) = V(i+y) - V(i)\,. \end{array}$$

Finally, $e^{\lambda[-M(i)+H_1(i)\wedge H_2(i,n)]} \leq e^{(K_1\wedge K_2+1)\lambda\eta n}$. By our choice of ε, we have for any integers k_1, k_2, ℓ_1, ℓ_2,

$$\begin{aligned} \mathbb{P}\left(K_1 = k_1,\, L_1 = \ell_1\right) &\leq \mathbb{P}\Big(V(\ell_1\lfloor \varepsilon n\rfloor) \in [-(k_1+2)\eta n, -(k_1-1)\eta n]\Big)\,, \\ \mathbb{P}\left(K_2 = k_2,\, L_2 = \ell_2\right) &\leq \mathbb{P}\Big(V\left(\ell_2\lfloor \varepsilon n\rfloor\right) \in [(k_2-1)\eta n, (k_2+2)\eta n]\Big)\,. \end{aligned}$$

By Cramér's theorem (see [4] for example),

$$\begin{aligned} \mathbb{P}\Big(V(\ell_1\lfloor \varepsilon n\rfloor) \in [-(k_1+2)\eta n, -(k_1-1)\eta n]\Big) &\lesssim \exp\Big(-\ell_1\lfloor \varepsilon n\rfloor(I(-x_1) - \lambda\eta)\Big) \\ \mathbb{P}\Big(V(\ell_2\lfloor \varepsilon n\rfloor) \in [(k_2-1)\eta n, (k_2+2)\eta n]\Big) &\lesssim \exp\Big(-\ell_2\lfloor \varepsilon n\rfloor(I(x_2) - \lambda\eta)\Big) \end{aligned}$$

if $-x_1$ is the point of $\left[\frac{-(k_1+2)\eta n}{\ell_1\lfloor \varepsilon n\rfloor}, \frac{-(k_1-1)\eta n}{\ell_1\lfloor \varepsilon n\rfloor}\right]$ where I reaches the minimum on this interval, and x_2 is the equivalent in $\left[\frac{(k_2-1)\eta n}{\ell_2\lfloor \varepsilon n\rfloor}, \frac{(k_2+2)\eta n}{\ell_2\lfloor \varepsilon n\rfloor}\right]$. It yields that

$$\begin{aligned} &\mathbb{E}\left[e^{\lambda[-M(i)+H_1(i)\wedge H_2(i,n)]}\right] \\ \lesssim\; &\max_{k_1,k_2,\ell_1,\ell_2\in D'} \exp\left(\left(k_1\wedge k_2\right)\lambda\eta n - I(-x_1)\ell_1\lfloor \varepsilon n\rfloor - I(x_2)\ell_2\lfloor \varepsilon n\rfloor + 3\lambda\eta n\right)\,, \end{aligned}$$

where D' is the (finite) set of all possible values of (K_1, K_2, L_1, L_2). We note that

$$\begin{aligned} &(k_1\wedge k_2)\lambda\eta n - I(-x_1)\ell_1\lfloor \varepsilon n\rfloor - I(x_2)\ell_2\lfloor \varepsilon n\rfloor \\ \leq\; &(x_1\ell_1\lfloor \varepsilon n\rfloor \wedge x_2\ell_2\lfloor \varepsilon n\rfloor)\lambda - I(-x_1)\ell_1\lfloor \varepsilon n\rfloor - I(x_2)\ell_2\lfloor \varepsilon n\rfloor + 3\lambda\eta n \\ \leq\; &(L(\lambda) + 3\lambda\eta)n \end{aligned}$$

by (8.1). Finally, $\mathbb{E}[e^{\lambda(-M(i)+H_1(i)\wedge H_2(i,n))}] \lesssim e^{n(L(\lambda)+6\lambda\eta)}$ so that, by equation (8.11), $m(n,\lambda) \lesssim e^{n(L(\lambda)+6\lambda\eta)}$. We let η tend to 0 to get that

$$\limsup_{n\to\infty} \frac{1}{n}\ln(m(n,\lambda)) \leq L(\lambda)\,.$$

Let $\lambda < \Lambda$. By definition of Λ and equation (8.4), it implies that $L(\lambda) < \frac{1}{q_1}$, so that we can find $r > q_1$ such that $\sum_{n\geq 0} m(n,\lambda) r^n < \infty$. It means that $\lambda \leq \lambda_c$. Consequently, $\Lambda \leq \lambda_c$. $\square$

8.2 Proof of Lemma 3.2

Fix x_1, $x_2 > 0$. Write

$$z_1 = \frac{x_2}{x_1 + x_2}, \quad z_2 = \frac{x_1}{x_1 + x_2}, \quad z = \frac{x_1 x_2}{x_1 + x_2}.$$

Let $a \geq 100$ and $n = n(a) := \lfloor \frac{\ln(a)}{z} \rfloor$. We have, by the strong Markov property, $P_\omega^0(T_{-1} \wedge T_n > a) \geq P_\omega^0(T_{\lfloor z_1 n \rfloor} < T_{-1}) P_\omega^{\lfloor z_1 n \rfloor}(T_{\lfloor z_1 n \rfloor} < T_{-1} \wedge T_n)^a$. It follows by (8.8), (8.9) and (8.10) that

$$\begin{aligned} p\,(n,a) &\gtrsim \mathbf{E}\Big[e^{-M(\lfloor z_1 n \rfloor)} \Big(1 - e^{-H_1(\lfloor z_1 n \rfloor) \wedge H_2(\lfloor z_1 n \rfloor, n)}\Big)^a\Big] \\ &\geq (1 - e^{-zn})^a \mathbf{P}\Big(V\left(\lfloor z_1 n \rfloor\right) < -zn,\, M\left(\lfloor z_1 n \rfloor\right) \leq 0\Big) \mathbf{P}\Big(V\left(\lfloor z_2 n \rfloor + 1\right) > zn\Big) \\ &\gtrsim \mathbf{P}\Big(V\left(\lfloor z_1 n \rfloor\right) < -zn,\, M\left(\lfloor z_1 n \rfloor\right) \leq 0\Big) \mathbf{P}\Big(V\left(\lfloor z_2 n \rfloor + 1\right) > zn\Big) \end{aligned}$$

by our choice of n. Let $k \geq 0$. Call τ the first time when the walk $(V(i))_{i \geq 0}$ reaches its maximum on $[0,k]$. Let $i \in [0,k]$ and for $0 \leq r \leq k-1$, $X_r := \ln(A_{\bar{r}})$ where $\bar{r} := i + r$ modulo k. We observe that

$$\begin{aligned} \mathbf{P}(V_k < -zn,\, \tau = i) &\leq \mathbf{P}(X_0 + \ldots + X_{k-1} < -zn,\, X_0 + \ldots + X_j \leq 0 \ \forall \ 0 \leq j \leq k-1) \\ &= \mathbf{P}(V_k < -zn,\, M_k \leq 0)\,. \end{aligned}$$

We obtain that $\mathbf{P}\left(V_k < -zn, M_k \leq 0\right) \geq \frac{1}{k+1} \mathbf{P}\left(V_k < -zn\right)$. Therefore, for any $\varepsilon > 0$,

$$\begin{aligned} p\,(n,a) &\gtrsim \mathbf{P}\Big(V\left(\lfloor z_1 n \rfloor\right) < -zn\Big) \mathbf{P}\Big(V\left(\lfloor z_2 n \rfloor + 1\right) > zn\Big) \\ &\gtrsim \exp\Big(n\left(-I(-x_1) z_1 - I(x_2) z_2 - 2\varepsilon\right)\Big) \end{aligned}$$

by Cramér's theorem. It yields that

$$\begin{aligned} \liminf_{a \to \infty} \left\{ \sup_{\ell \geq 0} \frac{\ln(q_1^\ell p\,(\ell, a))}{\ln(a)} \right\} &\geq \liminf_{a \to \infty} \frac{\ln(q_1^n p\,(n,a))}{\ln(a)} \\ &\geq \frac{\ln(q_1) - I(-x_1) z_1 - I(x_2) z_2 - 2\varepsilon}{z}\,. \end{aligned}$$

Finally, by (8.2) and (8.5),

$$\liminf_{a \to \infty} \left\{ \sup_{n \geq 0} \frac{\ln(q_1^n p(n,a))}{\ln(a)} \right\} \geq L' = -\Lambda\,. \square$$

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